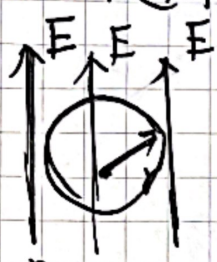
Mar. 5th. 2025. Topological Insulators & Superconductors.

1. Basis of topology.

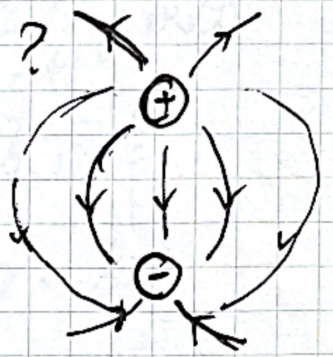
随着 $\vec{r}$ 转一圈 $\vec{E}(\vec{r})$ 绕n圈？



均强：0圈 ↑⇉↑



点电荷：1圈 or 0圈



偶极矩：2圈 or 1圈 or 0圈.

2. Berry Phase

$$H(R(t))|\psi(t)\rangle = i\partial_t|\psi(t)\rangle$$

since $H(t)$ is the same form at $t=0$ and $t=T$. how does the wave function varies ? $H|n(0)\rangle = e^{i0}$ to $H e^{-i\theta(t)}|n(t)\rangle$ $t=T$

$$|\psi(t)\rangle = e^{-i\theta(t)}|n_\alpha(t)\rangle$$

绝热近似下，$H(R(t),t)$ 1取决于R不同t不同R不同基态，2 $n_\alpha \to R_{R_t}$ 不仅改变了n，还增一相位.

$$\Rightarrow \hbar\partial_t\theta(t)|n_\alpha(t)\rangle + i\hbar\frac{d}{dt}|n_\alpha(t)\rangle = E_n(t)|n(t)\rangle$$

$E_n(t)$: $H(t)$ 本征值.

$$\Rightarrow E_n(t) = \hbar\partial_t\theta(t) + i\hbar\langle n(t)|\partial_t|n(t)\rangle$$

$$\theta(t) = \underbrace{\frac{1}{\hbar}\int_0^t E_n(t')dt'}_{\text{Dynamic Phase - trival}} - i\underbrace{\int_0^t\langle n(t')|\partial_{t'}|n(t')\rangle dt'}_{\text{Berry Phase.}}$$

我们假设 $|n(t)\rangle$ 随t周期变化 at 周期T. 比如

① 中绕一圈回来，所求的 $\theta(T)$：

$$\gamma_n = i\int_0^T \langle n_R(t) | \partial_{t'} | n_R(t') \rangle dt'$$

$$= i\int_0^T \langle n(R) | \nabla_R | n(R) \rangle \frac{dR}{dt'} dt'$$

t 从 0→T R从0→R 沿某条路径闭合.

$$= i \oint_{\text{Route闭合}} \langle n(R) | \vec{\nabla}_R | n(R) \rangle \cdot d\vec{R}$$

Stokes 定理



$$(\partial_{R_x}^{\rightarrow} \cdot \; d\vec{R}_x)$$

$$= i\iint_S d\vec{S}_R \cdot \vec{\nabla}_R \times \langle n(R) | \vec{\nabla}_R | n(R) \rangle$$

$$d\vec{S}_R \cdot (\partial R_x \cdot \partial R_y \cdot \partial R_z) \times (\partial R_x \cdot \partial R_y \cdot \partial R_z)$$

类比 $\oint \vec{E} \cdot d\vec{S}$ = 电通

$\gamma_n = \iint_S d\vec{S}_R \cdot (\cdots)$ "波通"

电通并非 invariant. 但闭合曲面 $\oiint_S \vec{E} \cdot d\vec{S} = Q$

Q是一个 invariant. 不论 E 多么复杂.

$\iint_S = \gamma_n \Rightarrow \oiint_S$ 是否 invariant ?

$$\gamma_n \to \gamma = -i \oiint \vec{\nabla}_k \times \langle k | \vec{\nabla}_k | k \rangle \cdot d\vec{S}_k = N \cdot 2\pi$$

Q form E

"N" form $(\nabla_k \cdots)$ named "chiral charge"

比如 Weyl 半金属对费米面积分. chiral → Weyl charge.

拓扑绝缘体 $(\cdots)$ 积分, $\mathbb{Z}_2$ (+1, -1)

3. In which form can certain Hamiltonian form a topological material & phase Non-Zero ?

Berry

$$\mathcal{H} = \vec{d}\cdot\vec{\delta} = \begin{pmatrix} d_z & dx - idy \\ dx + idy & -d_z \end{pmatrix}$$

$(\vec{B})$

eigenvalue: $\pm|d|$

eigenvector: $\frac{1}{\sqrt{2|d|(|d|-d_z)}}\begin{pmatrix} dx-idy \\ |d|-d_z \end{pmatrix}$

$$\frac{1}{\sqrt{2|d|(|d|+d_z)}}\begin{pmatrix} dx-idy \\ -|d|+d_z \end{pmatrix}$$

calculate: $\vec{\nabla}_k \times \langle n|\vec{\nabla}_k|n\rangle$ at Γ-θ-ϕ.

$$\begin{cases} dx = d\sin\theta\cos\phi \\ dy = d\sin\theta\sin\phi \\ dz = d\cos\theta \end{cases}$$

$\varphi_+ : \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$

$\varphi_- : \begin{pmatrix} e^{-i\phi}\sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$

$$\partial_\theta \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} = \begin{pmatrix} -\sin\theta/2\cdot\frac{1}{2} \\ e^{i\phi}\cos\theta/2\cdot\frac{1}{2} \end{pmatrix}$$

$$\partial_\phi \begin{pmatrix} e^{-i\phi}\cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} = \begin{pmatrix} 0 \\ ie^{i\phi}\sin\theta/2 \end{pmatrix}$$

$$\partial_\theta \begin{pmatrix} e^{-i\phi}\sin\theta/2 \\ -\cos\theta/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-i\phi}\cos\theta/2 \\ \frac{1}{2}\sin\theta/2 \end{pmatrix}$$

$$\partial_\phi \begin{pmatrix} e^{-i\phi}\sin\theta/2 \\ -\cos\theta/2 \end{pmatrix} = \begin{pmatrix} -ie^{-i\phi}\sin\theta/2 \\ 0 \end{pmatrix}$$

$\nabla_R = \nabla_r \hat{r} + \frac{1}{R}\nabla_\theta \hat{\theta} + \frac{1}{R\sin\theta}\nabla_\phi \hat{\varphi}$

$[\varphi \nabla_r \varphi] = 0.$

$$\hat{\theta}\cdot(\cos\theta/2\cdot e^{-i\phi}\sin\theta/2)\begin{pmatrix} -\sin\theta/2\cdot\frac{1}{2} \\ e^{i\phi}\cos\theta/2\cdot\frac{1}{2} \end{pmatrix} = -\frac{1}{4}\sin\theta + \frac{1}{4}\sin\theta = 0$$

$\Rightarrow \langle \varphi | \nabla_\varphi | \varphi \rangle \cdot \frac{1}{R\sin\theta}$

$= i \cdot \frac{\sin^2\frac{\theta}{2}}{R\sin\theta} \hat{e}_\varphi$ $\quad \vec{\nabla}_R \times \vec{F} = \frac{1}{R\sin\theta} \cdot \left(\frac{\partial(\sin\theta F_\varphi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi} \right) \hat{e}_r$

Step 2. $\left(\nabla_r \cdot \frac{1}{R}\nabla_\theta \cdot \frac{1}{R\sin\theta}\nabla_\varphi \right) \times \left(0 \cdot 0 \cdot i\frac{\sin^2\theta/2}{R\sin\theta} \right)$

$= \hat{r} \cdot \left(\frac{1}{R}\nabla_\theta \, i\frac{\sin^2\theta/2}{R\sin\theta} \right) - \hat{\theta} \cdot (\nabla_r (\cdots))$ $\quad (\nabla_r(\cdots) = 0)$

$= \hat{r} \cdot \frac{1}{2R^2} \cdot [\nabla_\theta \cdot \sin\theta/2] = \frac{\cos^2\theta}{4R^2} \hat{r}$

Step 3 $\therefore \iint_S \hat{r} \cdot \frac{\cos^2\theta}{2R^2} dS$  $\frac{i}{2R^2}$

(θ, ϕ) 绕一圈闭合 $\therefore -\frac{1}{2R^2} \iint_S \hat{r} \cdot d\vec{S}$

dS 沿 r 方向，就是球面.

$= -\frac{1}{2R^2} \iint_S dS_R$  $= -\frac{1}{2}\Omega$

$\iint_S \frac{1}{R^2} dS_R \rightarrow$ 立体角. 闭合曲面立体角为 4π.

$\Rightarrow \gamma = \oint_S = 2\pi \cdot 1$

Berry curvature: $\left(\frac{1}{2R^2} \hat{r} \right)$ chiral charge $\frac{1}{2}$ at $K=0$.

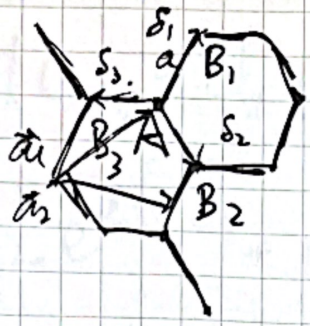
$E_\pm = \pm|d| = \pm|k|$

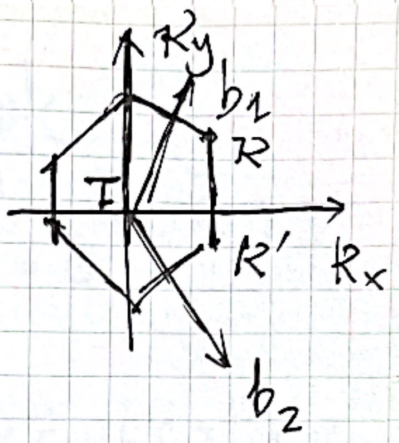
线性色散有非零 Berry Phase. $\oint E \cdot dS_R$

↑ θ $\quad \frac{1}{2}$ $\quad \rightarrow \frac{1\hbar|}{\gamma^3}$

Mar. 12th.

Graphene:





$\vec{\delta}_1: \frac{1}{2}a(1.\sqrt{3})$

$\vec{\delta}_2: \frac{1}{2}a(1.-\sqrt{3})$

$\vec{\delta}_3: a(-1.0)$

$K: \frac{2\pi}{3a}(1.\frac{1}{\sqrt{3}})$

$K': \frac{2\pi}{3a}(1.-\frac{1}{\sqrt{3}})$

$\vec{a}_1: \frac{a}{2}(3.\sqrt{3})$

$\vec{a}_2: \frac{a}{2}(3.-\sqrt{3})$

$\vec{b}_1: \frac{2\pi}{3a}(1.\sqrt{3})$

$\vec{b}_2: \frac{2\pi}{3a}(1.-\sqrt{3})$

Hamiltonian:

$$H = \sum_{i(A\&B)} U_A C_{iA}^+ C_{iA} + \sum_{iB} U_B C_{iB}^+ C_{iB}.$$
$$+ \sum_{iA} t_{B_1}\left[C_{iA}^+ C_{iB_1} + C_{iB_1}^+ C_{iA}\right]$$
$$+ \sum_{i} t_2\left[C_{iA} C_{iB_2} + C_{iB_2}^+ C_{iA}\right]$$
$$+ \sum_{i} t_3\left[C_{iA}^+ C_{iB_3} + C_{iB_3}^+ C_{iA}\right]$$

Let: $t_1 = t_2 = t_3$.

Fourier: $C_{Ai} = \sum_K C_{AK} e^{i\vec{k}\cdot\vec{r}_i}$

$$\sum_i U_1 C_A^+ C_A = U_1 \sum_i \sum_K C_{AK}^+ e^{-i\vec{k}\cdot\vec{r}_i} \sum_{K'} e^{i\vec{k}'\cdot\vec{r}_i} C_{AK}$$
$$= U_1 \sum_K C_{AK}^+ C_{AK}$$
$$\sum_i C_{Ai}^+ C_{B1i} = \sum_K C_{AK}^+ C_{BK} e^{i\vec{k}\cdot(\vec{r}_A - \vec{r}_{B1})}$$
$$= \sum_K C_{AK}^+ C_{BK} e^{i\vec{k}\cdot\vec{\delta}_1}$$

$$H = \sum_K U_1 C_{AK}^\dagger C_{AK} + U_2 C_{BK}^\dagger C_{BK}$$
$$+ t\, C_{AK}^\dagger C_{BK} \left[e^{ik\cdot\vec{\delta}_1} + e^{ik\cdot\vec{\delta}_2} + e^{ik\cdot\vec{\delta}_3} \right]$$
$$+ h.c.$$
$$= \sum_K \begin{pmatrix} C_{AK}^\dagger & C_{BK}^\dagger \end{pmatrix} \begin{pmatrix} U_1 & t\left[e^{ik\cdot\vec{\delta}_1} + e^{ik\cdot\vec{\delta}_2} + e^{ik\cdot\vec{\delta}_3}\right] \\ h.c. & U_2 \end{pmatrix} \begin{pmatrix} C_{AK} \\ C_{BK} \end{pmatrix}$$

$$H(k) = \begin{pmatrix} U_1 & t \sum_i \left(\cos(k\cdot\delta_i) + i\sin(k\cdot\delta_i)\right) \\ h.c. & U_2 \end{pmatrix}$$

$H(k) \Leftrightarrow \vec{d}\cdot\vec{\sigma} + \varepsilon$ where: (Let $U_1 = \varepsilon + m$, $U_2 = \varepsilon - m$).

where: $d_x = t \sum_i \cos(k\cdot\delta_i)$

$d_y = t \sum_i \sin(k\cdot\delta_i)$

$d_z = m$

$E_\pm^{(k)} = \sqrt{d_x^2 + d_y^2 + d_z^2}\ (\pm) + \varepsilon$.

Graphene: δ_i: $U_1 = U_2$, $m = 0$.

$$h(k) = \begin{pmatrix} 0 & t\sum_i e^{ik\cdot\delta_i} \\ h.c. & 0 \end{pmatrix}$$

where $E(k) = 0$ 费米面处 (源发处).

$$= \begin{pmatrix} C_{AK}^\dagger & C_{BK}^\dagger e^{-ik\cdot\delta_3} \end{pmatrix} \begin{pmatrix} 0 & e^{ik\cdot\vec{a}_1} + e^{ik\cdot\vec{a}_2} + 1 \\ h.c. & 0 \end{pmatrix} \begin{pmatrix} C_{AK} \\ C_{BK} e^{ik\cdot\delta_3} \end{pmatrix}$$

Find: $e^{ik\cdot a_1} + e^{ik\cdot a_2} + 1 = 0$.

Solution: $K_1 = K$, $K_2 = K'$

$$\mathcal{H}(k)_{K\to(K,K')} = \frac{3}{2}a\begin{pmatrix} 0 & \Delta k_y - i\Delta k_x \\ \Delta k_y + i\Delta k_x & 0 \end{pmatrix}$$

if $U_1 \neq U_2$, $m \neq 0$ (BN) 氮化硼.

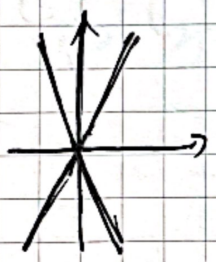
$$: \frac{3}{2}\begin{pmatrix} m & \Delta k_y - i\Delta k_x \\ \Delta k_y + i\Delta k_x & -m \end{pmatrix}$$

$$E = \pm\sqrt{\Delta k_x^2 + \Delta k_y^2 + m^2}$$

$\frac{1}{2}\frac{2k_x}{\sqrt{k_x^2+k_y^2+m^2}}$, 0

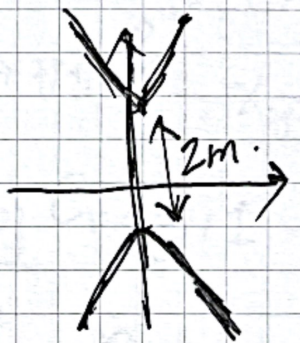
$\frac{\partial E}{\partial k_x}\Big|_{k_y=0}$

$m=0$



massless dirac fermion

$m \neq 0$

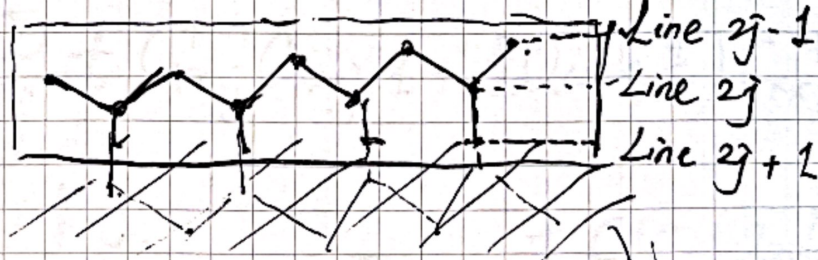


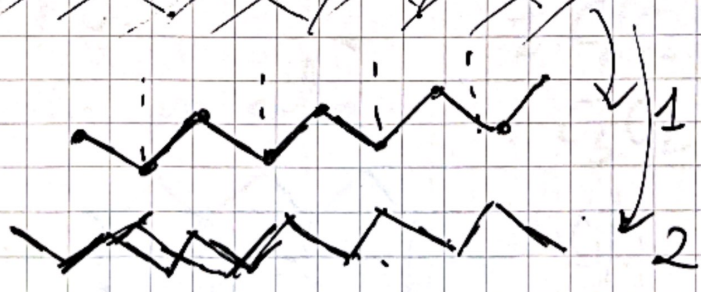
mass $2m$ dirac fermion.

gap ↔ gapless transition ⇄ mass / massless transition.

Edge state: x方向无限长, y方向有限宽带:

可沿x方向做Fourier, y方向对角化.





Graphene $U_1 = U_2 = 0$ choose Line $2j-1,\ 2j,\ 2j+1$.

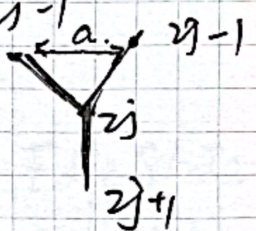
做 x 的 Fourier

$$C_{xy} = \frac{1}{N_x}\sum_{k_x} e^{-ik_x\cdot x} C_{k_x\cdot y}$$

$$H = \sum t\, C_{2j}^{+} C_{2j+1} + t\, C_{2j}^{+} C_{2j-1} + h.c.$$

1个 $2j+1$　　2个 $2j-1$

$$+ U\sum_i n_{i(j\cdot 2j-1\cdot 2j+1)} \quad (U=0)$$



Fourier:

$$H' \Rightarrow C_{2j}^{+} C_{2j+1} = C_{k_x 2j} C_{k_x 2j+1} e^{ik_x(X_{2j+1}-X_{2j})}$$

$X_{2j+1} = X_{2j}$

$$= \sum_{k_x\cdot j} C_{k_x\cdot 2j}^{+} C_{k_x 2j+1}$$

$$\Rightarrow C_{2j}^{+} C_{2j-1}\ \text{两个} = \sum_{k_x\cdot j} C_{k_x\cdot 2j}^{+} C_{k_x 2j-1} e^{ik_x\cdot\frac{a}{2}\cdot} + \sum_{k_x\cdot j} C_{k_x\cdot 2j}^{+} C_{k_x 2j-1} e^{ik_x\cdot -\frac{a}{2}\cdot}$$

$$= \sum_{k_x j} C_{k_x\cdot 2j}^{+} C_{k_x 2j-1}\left[2\cos k_x\tfrac{a}{2}\right]$$

$$H' = \sum_{\substack{k_x \\ j}} t\, C_{k_x 2j}^{+} C_{k_x 2j+1} + t(2\cos k_x\tfrac{a}{2})\, C_{k_x 2j}^{+} C_{k_x 2j-1}\cdot$$

$+ h.c.$

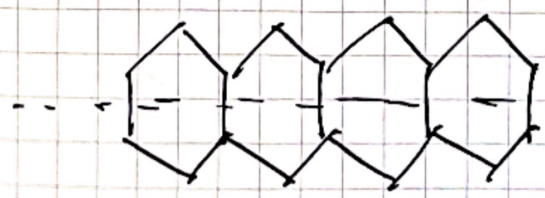
The Lattice we solve:

$t\, C_2^{+} C_1 (2\cos k_x \frac{a}{2})$

$+ C_3^{+} C_2$

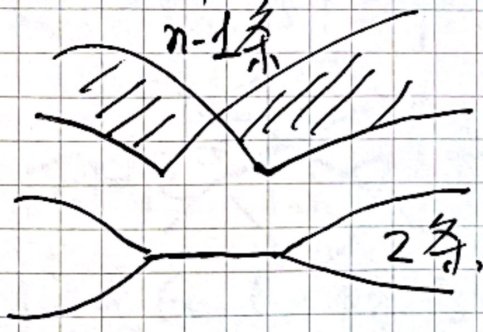
$+ C_4^{+} C_3 (2\cos k_x \frac{a}{2})$

$+ h.c.$

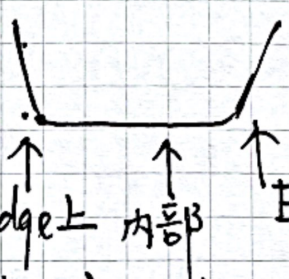


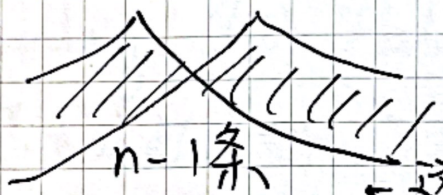
$$h = \begin{pmatrix} 0 & 2\cos k_x a/2 & 0 & 0 \\ 2\cos k_x \frac{a}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & 2\cos k_x a/2 \\ 0 & 0 & 2\cos k_x \frac{a}{2} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}$$

Solve: 2n条能带.



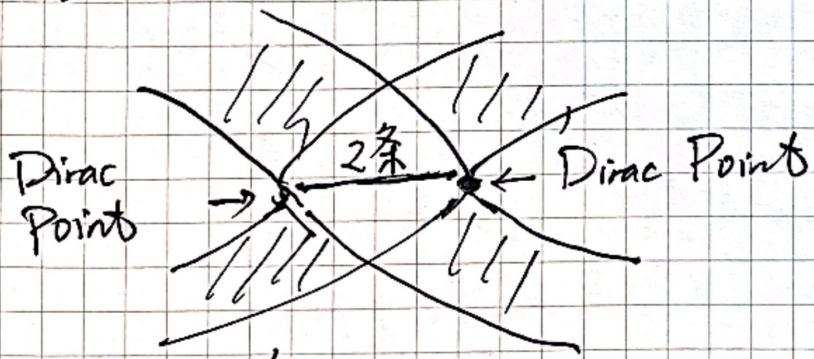
这两条波函数 Edge State. $\varphi(x)$



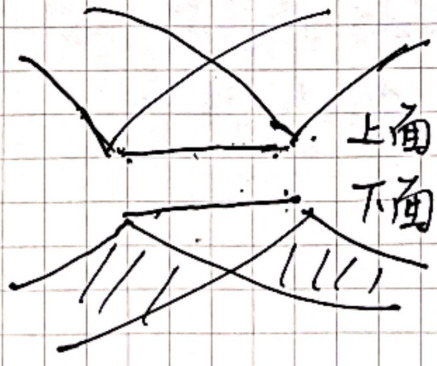


这些波函数分布均匀

当 $n \to \infty$



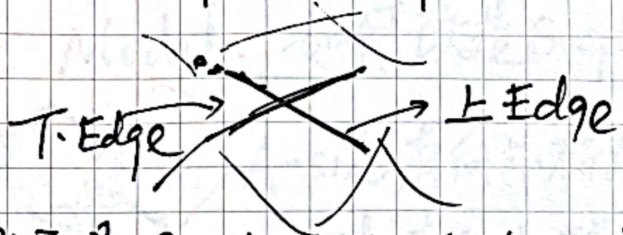
而m和λ_{so}引入 Break Inversion symmetry.



为什么不能相交？
两类边缘态何义？

※ 如果是一个拓扑材料，必有一个 Gapped Bulk 与 Gapless Edge state，而非拓扑的材料没有能保证 Gappless 的性质。

氮化硼虽然有 Gapped Bulk，而 Edge State 不是 Gappless 的，或者只能因边缘形状改变而变得 Gappless 并非100%为 Gappless 的，如果两个 Edge state 有如下的两个能带



其才必是 Gappless 的，无论边缘形状，无论如何做 Inverse Breaking，名为 Haldane Gappless。

石墨烯，亦非拓扑的，K取遍所有K值，d无法取遍所有d值，($d_z=m$)，不够好的 $H=d\cdot\delta$ 如果改造 d_z 亦使之用(cos, sin)构造取遍R才够好。

$d(k)$ K取遍R K space 是一个闭合曲面

因此 $d(k)$ d space 亦是一个闭合曲面

而由于 $d_z=m$ 不够好，该曲面不包括原点。

而 chiral charge 处于原点。

因此 $d(k)$ 不包括 chiral charge 为零无 Berry Phase

非拓扑的。

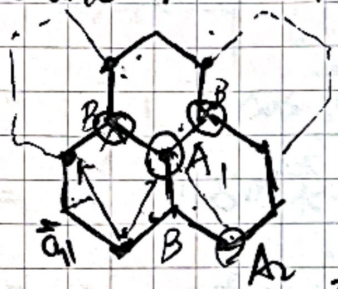
Mar. 19th.

Topological material:

① Hamiltonian 能写作 $H = d \cdot \sigma$ Dirac Fermion 形式.

② 自变量取遍 $\vec{k}$. d 可以取到原点的 k-space 内. 因此. 积分 Berry Phase 才非零. chiral charge.

Haldane Model. 如何改进 Graphene 使之包括原点



A-site 有向外磁性. 周期性磁场

B-site 有向内磁性. 反铁磁

Hamiltonian 加入一项. $\sum_{ij} c_i^\dagger c_j e^{i\phi_{ij}}$ $\phi_{ij} = \frac{e}{\hbar}\int_i^j \vec{A}\cdot d\vec{r}$

$$H = \sum_i \mu_i n_i + t_1 \sum_{A\to B} c_i^\dagger c_j + t_2 \sum_{\substack{A\to A \\ B\to B}} c_i^\dagger c_j e^{i\phi_{ij}}$$

$A\to B$ 忽略相位. nearest-neighbor. 和 second-nearest neighbor.

A.B 自旋相反. 磁场对 A.B 相位相消

A.A 自旋相同. 有 (ϕ_{ij}) 相位作用

$$\Rightarrow t_2 \left[e^{i\phi_{B_A}} c_{A_1}^\dagger c_{A_2} + e^{-i\phi_{B_B}} c_{B_1}^\dagger c_{B_2} + e^{i\phi_{\cdots}} + h.c. \right]$$

Fourier:

$$e^{-i\phi} c_{A_1}^\dagger c_{A_2} = e^{-i\phi} \sum_k c_{kA}^\dagger c_{kA} e^{i\vec{k}\cdot(\vec{r}_A - \vec{r}_{A'})}$$

$$= e^{-i\phi} \sum_k^K c_{kA}^\dagger c_{kA} \left[e^{i\vec{k}\cdot\vec{a}_1} + e^{i\vec{k}\cdot\vec{a}_2} + e^{i\vec{k}\cdot(-\vec{a}_1 - \vec{a}_2)} + h.c. \right]$$

$e^{-i\phi}$: A.ϕ 相同

A_1 有 a 个 A_2. b

ϕ_{ij} 六个相同



$e^{-i\phi} C_A^\dagger C_{A'} + h.c. \Rightarrow$

$e^{-i\phi} \cdot e^{i\vec{k}\cdot\vec{a}_1} + e^{i\phi} e^{-i\vec{k}\cdot\vec{a}_1} = 2\cos\phi\cos(\vec{k}\cdot\vec{a}_1) + 2\sin\phi\cos(\vec{k}\cdot\vec{a}_1)$

因此. If we define.. Fourier 后全有:

$$\begin{pmatrix} O_{kA}^\dagger \\ C_{kB}^\dagger \end{pmatrix}^T \begin{pmatrix} d_0+d_3 & d_1+id_2 \\ d_1-id_2 & d_0-d_3 \end{pmatrix} \begin{pmatrix} C_{kA} \\ C_{kB} \end{pmatrix}$$

$$= \vec{d}\cdot\vec{\sigma}$$

$$d_0 = 2t_2\cos\phi\left[\cos\vec{k}\cdot\vec{a}_1 + \cos\vec{k}\cdot\vec{a}_2 + \cos\vec{k}\cdot(\vec{a}_1+\vec{a}_2)\right] + \varepsilon$$

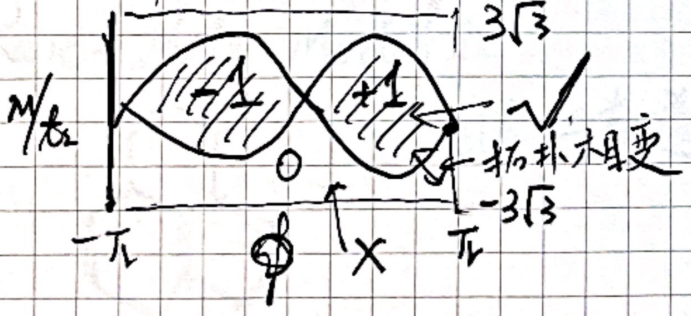
$$d_1 = t_1\left[\cos\vec{k}\cdot\vec{a}_1 + \cos\vec{k}\cdot\vec{a}_2 + 1\right]$$

$$d_2 = t_1\left[\sin\vec{k}\cdot\vec{a}_1 + \sin\vec{k}\cdot\vec{a}_2 + 1\right]$$

$$d_3 = m + 2t_2\sin\phi\left[\sin\vec{k}\cdot\vec{a}_1 - \sin\vec{k}\cdot\vec{a}_2 + \sin\vec{k}(\vec{a}_1+\vec{a}_2)\right]$$

$H = d_0 + (d_1, d_2, d_3)\cdot\vec{\sigma}$ $d_3=0$ 可成立.

取参数 ϕ. t_2. 使 d_3 包括奇点. 调节材料使之产生拓扑相变



Band Inversion. Expand at $(K. K')$

$$h(K+\Delta k) = \frac{3}{2}\frac{t_1}{\hbar}(\Delta k_x\sigma_x - \Delta k_y\sigma_y) + (m - 3\sqrt{3}t_2\sin\phi)\sigma_z$$

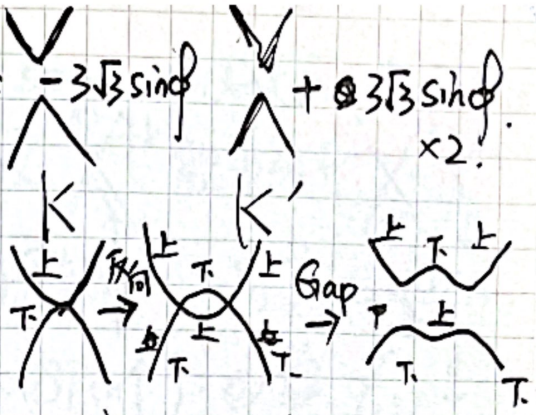
$$h(K'+\Delta k) = \frac{3}{2}\frac{t_1}{\hbar}(-\Delta k_x\sigma_x - \Delta k_y\sigma_y) + (m + 3\sqrt{3}t_2\sin\phi)\sigma_z$$

Let $m=0$: Graphene. Let $\sin\phi=0$. $\Rightarrow$ Dirac 锥 ✗

Let $m=0$. $\sin\phi \neq 0$. Open Gap $2\times -3\sqrt{3}\sin\phi$ $+3\sqrt{3}\sin\phi$ $\times 2$.

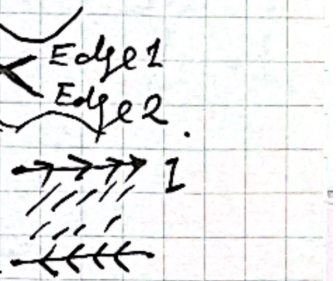
Gap正负体现能带反转

(+-) Haldane Mass Eig2 Eig1 / Eig2 Eig1



此时再次去画边缘态($E=0$) state.

此时群速度 $\frac{d}{dk}E(k)$ Edge1 $V=(+X)$ Edge2 $V=(-X)$



因此上边界电子只能向前. 下边界电子只能向后

Quantum Hall中边界电流系一半的Lorentz圈，此Haldane Model中系群速相反 Chern number (N).

实际就是边缘态条数

总的磁场为零，但内部存在反铁磁

Kane-Mele Model:

自旋-轨道耦合: $\frac{e}{mc}\vec{s}\cdot\vec{v}=\frac{e}{mc}\vec{s}\cdot(\vec{p}\times\vec{\delta})$



$$H=\underbrace{\sum_{AB\sigma} U_i c_{i\sigma}^{+} c_{i\sigma}}_{+h.c.\ 1}+\underbrace{t\sum_{\substack{A\to B\\ \delta\sigma}} c_{i\sigma}^{+} c_{j\sigma}}_{2}+\underbrace{i\lambda_R \sum_{\substack{A\to B\\ \sigma}} c_i^{+}\left(\vec{s}\times\vec{r}_{ij}\right)_z c_j}_{4}+\underbrace{i\lambda_{so}\sum_{\substack{A\to A'\\ B\to B'}} \nu_{ij} c_i^{+} s_z c_j}_{3}$$

σ: $\uparrow$. $\downarrow$

$\nu_{ij}=\pm 1$

前两项 Fourier. 在 $(c_{A\uparrow}^{+}\ c_{B\uparrow}^{+}\ c_{A\downarrow}^{+}\ c_{B\downarrow}^{+})$ 的 Basis 下面.

$$\begin{pmatrix} 0 & \sum t e^{i\vec{k}\cdot\vec{\delta}_i} & & \\ \sum t e^{-i\vec{k}\cdot\vec{\delta}_i} & 0 & & 0 \\ & & 0 & \sum t e^{i\vec{k}\cdot\vec{\delta}_i} \\ & 0 & \sum t e^{-i\vec{k}\cdot\vec{\delta}_i} & 0 \end{pmatrix}$$

$H(k)$ $\begin{matrix} C_{A\uparrow}^+ \\ C_{B\uparrow}^+ \\ C_{A\downarrow}^+ \\ C_{B\downarrow}^+ \end{matrix}\begin{pmatrix} 1\,2\,3 & 4 \\ 4 & 1\,2\,3 \end{pmatrix}$

spin-orbital coupling 使 ↑↓ 能带分开

$H_{1.2.3.}$ 就是 Haldane model.

分开能带 (↑,↓) 产生 ~~Gapless~~.

H_4 的引入 Calculate Berry Phase: (Spin↑): 总的 $Z_2 = 1$

总的 Berry Phase 为 0. 为 $1+(-1)=0$. (Spin↓): $Z_2 = -1$.

$$H_3 = i\lambda_{so}\sum \nu_{ij} C_i^+ S^z C_j$$

$$= i\lambda_{so}\sum \nu_{ij} C_{A\sigma}^+ S^z C_{A'\sigma} + i\lambda_o \sum \nu_{ij} C_{B\sigma}^+ S^z C_{B'\sigma}.$$

S^z: (pauli z)

$$= C_{A\uparrow}^+ C_{A'\uparrow} - C_{A\downarrow}^+ C_{A'\downarrow} \Rightarrow i\lambda_{so}[C_{A\uparrow}^+ C_{A\uparrow} - C_{B\uparrow}^+ C_{B\uparrow}]$$

$$- i\lambda_{so}[C_{A\downarrow}^+ C_{A'\downarrow} - C_{B\downarrow}^+ C_{B\downarrow}]$$

Haldane model 的相位 $e^{i\nu_{ij}} = i$

Fourier: $i\lambda_{so}[C_{A\uparrow}^+ C_{A\uparrow} - C_{B\uparrow}^+ C_{B\uparrow}]$

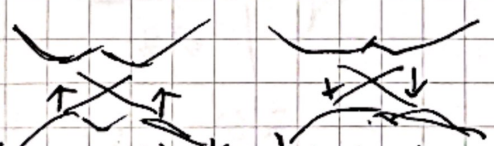
$$= \begin{pmatrix} C_{Ak\uparrow}^+ & C_{Bk\uparrow}^+ \end{pmatrix}\begin{pmatrix} \varepsilon+d_3 & 0 \\ 0 & \varepsilon-d_3 \end{pmatrix}\begin{pmatrix} C_{Ak\uparrow} \\ C_{Bk\uparrow} \end{pmatrix}$$

$\varepsilon = 2\lambda_{so}\cos\pi/2 = 0$

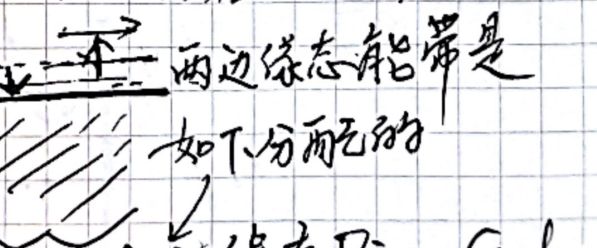
$d_3 = 2\lambda_{so}[\sin\vec{k}\cdot\vec{a}_1 + \sin\vec{k}\cdot(-\vec{a}_2) + \sin\vec{k}\cdot(-\vec{a}_1-\vec{a}_2)]$

$= ($ 4个 $)\left(\begin{array}{c|c} d_3\delta_z & \\ \hline & -d_3\delta_z \end{array}\right)($ 4个 $)$

Spin↑ 与 Spin↓ 皆是 Haldane: 每个边界上有两条边缘态: ↑与↓. 在边缘某一个上:



强拓扑绝缘体(3D)的表面态上有 Dirac 手性 $\vec{S}\cdot\vec{L}$



↑↓的 Dirac 锥



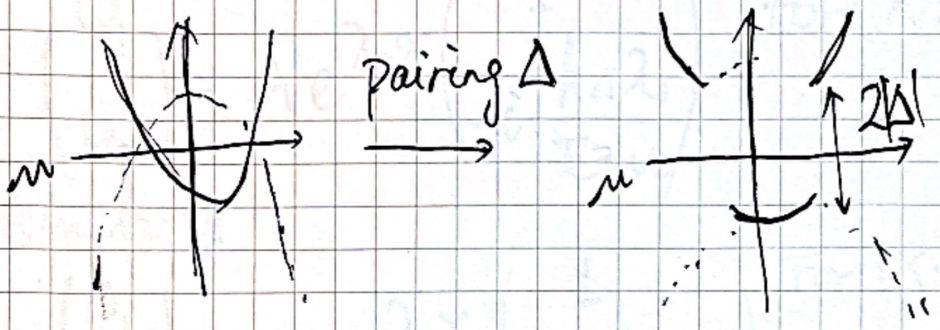
Topological Superconductor:

s-wave superconductor:

$$\mathcal{H} = c_p^\dagger \left(p^2/2m - \mu\right) c_p + \Delta c_p^\dagger c_{-p}^\dagger + h.c.$$

$$\mathcal{H}_{BdG} = \begin{pmatrix} c_p^\dagger & c_{-p} \end{pmatrix} \begin{pmatrix} p^2/2m - \mu & \Delta \\ \Delta^\dagger & -p^2/2m + \mu \end{pmatrix} \begin{pmatrix} c_p \\ c_p^\dagger \end{pmatrix}$$

$$E_\pm = \sqrt{\left(p^2/2m - \mu\right)^2 + |\Delta|^2}$$

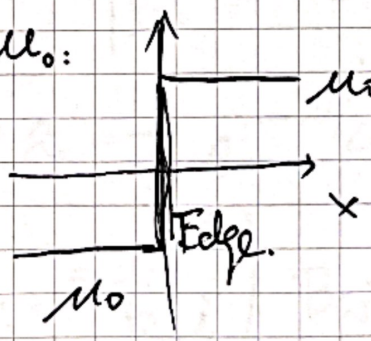


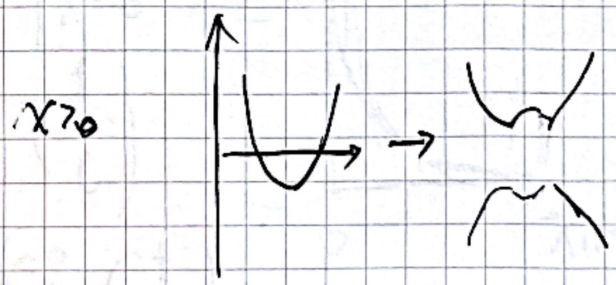
p-wave superconductor in 1-D.

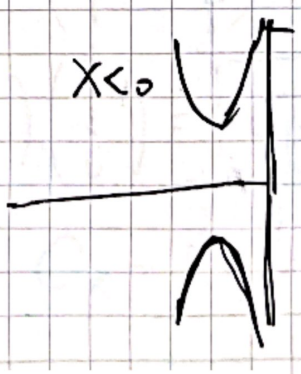
$$\mathcal{H}_{BdG} = \begin{pmatrix} p^2/2m - \mu & \Delta p \\ \Delta^* p & -p^2/2m + \mu \end{pmatrix} \begin{cases} \hat{p} = -i\hbar \frac{d}{dx} \\ m \to \infty \\ \Delta = \Delta^* \end{cases} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

半无限长纳米线. 有 majorana Zero modes.

取 μ_0:







Edge State 位于
绝缘-超导体边界.

Solving E: $-(E+u)\varphi_1 - i\hbar\Delta\frac{d}{dx}\varphi_2 = 0$.

$$i\hbar\Delta\frac{d}{dx}\varphi_1 + (u-E)\varphi_2 = 0.$$

we have: $(u_0^2 - E^2)\varphi_2 = \hbar^2\Delta^2\frac{d^2}{dx^2}\varphi_2$.

$$\Rightarrow \varphi_1 = -\frac{i\hbar\Delta}{E+u_0}\lambda_+ e^{\lambda_+ x} \quad a_+ = -\frac{\sqrt{u^2-E^2}}{\Delta\hbar}$$

$$C_p^+ \to \Psi^+(r) \qquad p \to \partial/\partial x$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = Ae^{\lambda_+ x}\begin{pmatrix}1\\ -i\frac{\hbar\Delta\lambda_+}{E+u}\end{pmatrix} \quad \text{For } (x>0)$$

similarly

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = B\cdot e^{\lambda_- x}\begin{pmatrix}1\\ -i\frac{\hbar\Delta\lambda_-}{E-u}\end{pmatrix} \quad \text{For } (x<0). \qquad \lambda_- = \frac{\sqrt{u^2-E^2}}{\Delta\hbar}$$

待定(A. B. E)

$$\varphi_1(x>0)\Big|_{x=0} = \varphi_1(x<0)\Big|_{x=0}.$$

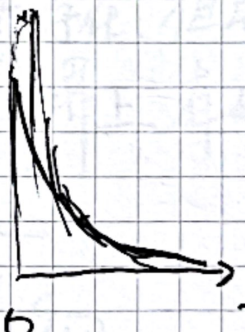
$$\varphi_2(x>0)\Big|_{x=0} = \varphi_2(x<0)\Big|_{x=0}$$

$C_p^+|0\rangle = |p\rangle$

$\varphi^+|0\rangle = |\varphi\rangle$

$\Rightarrow E = 0$

$$|\psi\rangle \approx \frac{1}{\sqrt{2}}e^{-u/\Delta\hbar x}\begin{pmatrix}1\\ i\end{pmatrix}$$



$$|\psi\rangle_{x=0} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{Basis: } (C_p^+ \; C_{-p}^+)$$

Eigen Basis

$$\langle p|\psi\rangle = (C_p \; C_{-p}^+)\begin{pmatrix}1\\ i\end{pmatrix} = \boxed{C_p + iC_{-p}^+}$$

$$\int\langle p|\varphi\rangle e^{ipx}dp = C_x - iC_x^+ = \gamma \quad \text{月} \quad \gamma^+ = \gamma$$

$$(e^{i\pi/4}\gamma)^+ = e^{-i\pi/4}\gamma$$

p-wave superconductor in 2-D

$$H_{BdG} = \begin{pmatrix} p^2/2m - \mu & \Delta(p_x + ip_y) \\ \Delta^*(p_x - ip_y) & -p^2/2m + \mu \end{pmatrix}$$

Kitaev: $\mathbb{Z}_2$.

p+ip: Chern. C = 1

$$= \begin{pmatrix} -\hbar^2/2m\left(\partial/\partial r^2 + \frac{1}{r}\partial/\partial r + \frac{1}{r^2}\partial^2/\partial\theta^2\right) - \mu, & \Delta e^{i\theta}\left(\partial/\partial r + \frac{1}{r}\partial/\partial\theta\right) \\ (\cdots)^\dagger & -(\cdots). \end{pmatrix}$$

$$\Rightarrow |\psi\rangle = g(r)\cdot\begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}$$

$$|\Gamma\rangle = g(r)\, e^{i\theta/2} c_{r\theta} + e^{-i\theta/2} c^\dagger_{r-\theta} |\,\rangle_{FS}.$$

$H = d\cdot\sigma$ $\quad d_x = \Delta p_x \quad d_y = \Delta p_y$

$d_z = \frac{p_x^2 + p_y^2}{2m} - \mu$ 可以闭合曲面包含原点

2-D Topological Superconductor 的 1-D 边界存在. 但非好的 majorana 定义. 前加磁场后的 0-D 边界上有如上的 $|\Gamma\rangle$ 波函数. 亦是 majorana.

Fu-Kane model: 低能 $\Leftrightarrow$ p+ip 2D.

① H_{3D} ~ surface: $v_F(\vec{p}\times\vec{s})\cdot\hat{e}_z = v_F(p_x\sigma_y - p_y\sigma_x)$ 为有手性 Dirac 锥

Topological Insulator

② S-wave SC: $[\Delta c^\dagger_{p\uparrow} c^\dagger_{-p\downarrow} + \Delta c_{p\uparrow} c_{-p\downarrow}]$

这样一个3D Toplogical Insulator的2D surface又是超导S-波的，在3-D TI表面上长出一层2D S-wave，比如其它材料，它自然又是3-D TI的2-D surface：实验上可行

$$H_S = \begin{pmatrix} \\ 4\times 1 \\ \\ \end{pmatrix}^{\dagger} \times \begin{pmatrix} 0 & -i\Delta S_y \\ i\Delta S_y & 0 \end{pmatrix}$$

$$\Rightarrow H_{FK}: \begin{pmatrix} v(P_x S_y - P_y S_x) - \mu & -i\Delta S_y \\ i\Delta S_y & -v(P_x S_y - P_y S_x) + \mu \end{pmatrix}$$

超导近邻放在S-wave超导体上长一个3D Toplogical Insulator的薄膜：

H_{FK} 微扰展开：极坐标下：

$$\begin{pmatrix} -\mu & Pe^{i\theta} & 0 & \Delta \\ Pe^{-i\theta} & -\mu & -\Delta & 0 \\ & \cdots & & \end{pmatrix}$$

$\Delta=0$ 对应 H_0，以 Δ^1 微扰代入 H_0

$$H_0: \chi_+: \frac{1}{\sqrt{2}}\begin{pmatrix} -ie^{i\theta} \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \chi_-: \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -ie^{i\theta} \\ 1 \end{pmatrix}$$

$$\delta H: \begin{pmatrix} -\mu & \delta P e^{i\theta} & 0 & \Delta \\ \delta P e^{-i\theta} & -\mu & -\Delta & 0 \\ & \cdots & & \end{pmatrix}$$

$$\langle \chi_+ | \delta H | \chi_+ \rangle = \delta p - \mu$$

$$\langle \chi_+ | \delta H | \chi_- \rangle = i\Delta e^{-i\theta}$$

$$\Rightarrow \delta H = \begin{pmatrix} \delta p - \mu & -i\Delta e^{i\theta} \\ i\Delta e^{i\theta} & -\delta p + \mu \end{pmatrix}$$

Δp

$p e^{i\theta} = p_x + i p_y$

$p\cos\theta + ip\sin\theta$

$$= \begin{pmatrix} \delta p - \mu & -i\Delta(p_x + i p_y) \\ i\Delta(p_x - i p_y) & -\delta p + \mu \end{pmatrix} \Leftrightarrow p+ip \text{ wave.}$$