虚时格林函数 (Imaginary-time Green's Function) (松原)

我们写关联函数，用迹，用相互作用绘景写出：

$$C_{AB}(t,t') = -\langle A(t)B(t')\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\hat{U}(0,t)\hat{A}(t)\hat{U}(t,t')\hat{B}(t')U(t',0)\right]$$

量子中介绍 $t \to -i\tau$ 可以求 Energy 基态，一激…… $\tau = it$

相互作用绘景：

$$\begin{cases} \hat{A}(\tau) = e^{\tau H_0} A\, e^{-\tau H_0} \\ \hat{U}(\tau,\tau_0) = e^{\tau H_0} e^{-(\tau-\tau')H} e^{-\tau H_0} \end{cases}$$

$$\partial_\tau \hat{U}(\tau,\tau') = e^{\tau H_0}(H_0 - H)e^{-(\tau-\tau')H}e^{-\tau' H_0} = -\hat{V}(\tau)\hat{U}(\tau,\tau')$$

solve: ODE, we have:

$$\hat{U}(\tau,\tau') = T_\tau \exp\left(-\int_{\tau'}^{\tau} d\tau_1 \hat{V}(\tau_1)\right)$$

T_τ：时序算符，

$$\tau > \tau' \quad T[A(\tau)B(\tau')] = A(\tau)B(\tau')$$

$$\tau \leq \tau' \quad T[A(\tau)B(\tau')] = B(\tau')A(\tau)$$

$$e^{-\beta H} = e^{-\beta H_0}\hat{U}(\beta,0) = e^{-\beta H_0} T_\tau \exp\left(-\int_0^\beta d\tau_1 V(\tau_1)\right).$$

$\langle \mathrm{Tr}(T_\tau[A(\tau)B(\tau')])\rangle \frac{1}{Z}$ 即 $\langle T_\tau(A(\tau)B(\tau'))\rangle$

求Trace这些随便换：$= \frac{1}{Z}\mathrm{Tr}\Big(T_\tau(\underbrace{U(\beta,0)}_{(\beta,\tau)(\tau,\tau')(\tau',0)} A(\tau)B(\tau'))\, \underbrace{\exp(-\beta H_0)}_{U(\beta,0)}\Big)$

Therefore:

$$\langle T_\tau(A(\tau)B(\tau'))\rangle = \frac{1}{Z}\frac{\mathrm{Tr}(e^{-\beta H_0}T_\tau(\hat{U}(\beta,0)A(\tau)B(\tau')))}{\mathrm{Tr}(e^{-\beta H_0}U(\beta,0))}$$

$$G'' \qquad \boxed{= \frac{\langle T_\tau \hat{U}(\beta,0)A(\tau)B(\tau')\rangle}{\langle \hat{U}(\beta,0)\rangle}}$$

$\hat{U}(\beta,0)$，β不是$(KT)^{-1}$？因为定义虚时$\tau = it$目的即让配分函数$e^{-\beta H}$与时间演化算符e^{-iHt}数学上一致，极大便于计算。

说明：求 Tr: $\rho = e^{-\beta H} = e^{-\beta H_0}\cdot e^{-\beta(H-H_0)}$，实时：$\hat{u} = e^{-i(\beta H-H_0)}$
少一个 i 求不了，且 $i\partial_t u = Hu$，若用虚时，$\partial_\tau u = Hu$ $\quad u = e^{-\beta H}$
可以直接求 $\rho = e^{-\beta H}$ 如上所述.

定义虚时（松原）格林函数为：

$$\mathcal{G}_{AB}(\tau,\tau') = -\langle T_\tau[A(\tau)B(\tau')]\rangle$$

+: Boson −: Fermion

$$T_\tau[A(\tau)B(\tau')] = \theta(\tau-\tau')A(\tau)B(\tau') \pm \theta(\tau'-\tau)B(\tau')A(\tau)$$

性质：

① $\mathcal{G}$ 仅依赖 $\tau-\tau'$ $\quad \mathcal{G}(\tau,\tau') = \mathcal{G}(\tau-\tau')$

$= \frac{1}{Z}\mathrm{Tr}[e^{-\beta H}e^{\tau H}Ae^{\tau H}e^{\tau' H}Be^{-\tau' H}]$ 求迹随便换

$= \frac{1}{Z}\mathrm{Tr}[e^{-\beta H}e^{(\tau-\tau')H}Ae^{-(\tau-\tau')H}B]$

$= \mathcal{G}_{AB}(\tau-\tau')$

② $\mathcal{G}_{AB}(\tau,\tau')$ 在 $\tau-\tau' \in (-\beta,\beta)$ 才收敛

$\exp((-\tau+\tau'+\beta)E_n) > 0$ 发散（Taylor 展取阶实，愈发发散）.

③ $\mathcal{G}_{AB}(\tau+\beta) = \pm\mathcal{G}_{AB}(\tau)$ for $\tau<0$

Fourier 变换.

$\tau \in (-\beta,\beta)$ $\quad \tau: \tau-\tau'$ 时间参数.

$$\mathcal{G}_{AB}(n) = \int_{-\beta}^{\beta} e^{\underbrace{i\pi n\tau/\beta}_{\omega}}\mathcal{G}_{AB}(\tau)\,d\tau$$

$$\mathcal{G}_{AB}(\tau+\beta) = \pm\mathcal{G}(\tau) \quad \begin{matrix}+:\text{Boson}\\ -:\text{Fermion}\end{matrix}$$

$$= \frac{1}{2}(1\pm e^{-i\pi n})\int_0^\beta d\tau\, e^{i\pi n\tau/\beta}\mathcal{G}_{AB}(\tau).$$

therefore we have:

$$G_{AB}(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau} G_{AB}(\tau) \qquad \omega_n = \frac{2n\pi}{\beta} \text{ for Boson} \qquad \omega_n = \frac{(2n+1)\pi}{\beta} \text{ for Fermion}$$

ω_n: 松原频率.

解释：τ 看似乎接上文类似一个实数？why we apply imaginary time Green's function？，Fourier: For Retard single-particle Green's function, $G^R(r,\omega) = \frac{1}{Z}\sum_{nn'} \frac{\langle n|A|n'\rangle\langle n'|B|n\rangle}{\omega + E_n - E_{n'} + i\eta}(e^{-\beta E_n} \pm e^{\beta E_{n'}})$; after calculation $G(i\omega_n) = \frac{1}{Z}\sum_{nn'} \frac{\langle n|A|n'\rangle\langle n'|B|n\rangle}{i\omega_n + E_n - E_{n'}}(e^{-\beta E_n} \pm e^{\beta E_{n'}})$, therefore, there's always $G(Z)$ in complex field $(i\omega_n)$: $i\omega_n \to \omega + i\eta$ exists in 巨正则系综, after G, analytic continuation exchange with G.

Actually, $G(Z)$ 分母: $Z + E_n - E_{n'}$ 解析 in 上、下复平面，在实轴上存在极点，我们之前考虑 G "性质" 时，一般其实在考虑 $G(Z)$ 性质，要 ω 成立，才有①.②.③，我们之前考虑者是实轴上 $G(x)$. 现在我们知道虚轴上 $G(i\omega)$ 有如上文所示诸多好性质，并且可以在复平面上利用围道积分求出，若知 $G(i\omega)$，解析延拓即知 $G^R(\omega)$ Retard Green's Function.

解析延拓：$f(z)$ analytic in $\mathbb{C}$, on $\mathbb{R}$. $f(x) = F(x)$. and $F(z)$ analytic in $\mathbb{C}$, Then, $f(z)$ can continuation in $\mathbb{C}$, name $F(z)$ is the analytic continuation of $f(x)$ in $\mathbb{C}$.

下面我们就利用 G，求解单粒子的 G：

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无相互作用粒子： $H_0=\sum_\nu \varepsilon_\nu c_\nu^\dagger c_\nu$

Heisenberg picture： $c_\nu(\tau)=e^{\tau H_0}c_\nu e^{-\tau H_0}=\exp(-\varepsilon_\nu\tau)c_\nu$

$c_\nu^\dagger(\tau)=e^{\tau H_0}c_\nu^\dagger e^{-\tau H_0}=\exp(\varepsilon_\nu\tau)c_\nu^\dagger$

$$\mathcal{G}_0(\nu,\tau-\tau')=-\langle T_\tau(c_\nu(\tau)c_\nu^\dagger(\tau'))\rangle$$

$$=-\left[\theta(\tau-\tau')\langle c_\nu c_\nu^\dagger\rangle(\pm)\,\theta(\tau'-\tau)\langle c_\nu^\dagger c_\nu\rangle\right]e^{-\varepsilon_\nu(\tau-\tau')}$$

比如费米子：

$$\mathcal{G}_{0.\mathrm{Fermion}}(\nu,\tau-\tau')=-\left[\theta(\tau-\tau')[1-n_F(\varepsilon_\nu)]-\theta(\tau'-\tau)\,n_F(\varepsilon_\nu)\right]e^{-\varepsilon_\nu(\tau-\tau')}.$$

Fourier Transform：

$$\mathcal{G}_{0\,\mathrm{fermion}}(\nu,ik_n)=\int_0^\beta d\tau\, e^{ik_n\tau}\,\mathcal{G}_{0F}(\nu,\tau)\qquad k_n=(2n+1)\frac{\pi}{\beta}$$

$\tau:\tau-\tau'>0\Rightarrow -(1-n_F(\varepsilon_\nu))\int_0^\beta d\tau\, e^{ik_n\tau}e^{-\varepsilon_\nu\tau}$

$=\dfrac{n_F(\varepsilon_\nu)-1}{ik_n-\varepsilon_\nu}\left(e^{ik_n\beta}e^{-\varepsilon_\nu\beta}-1\right)$ $\quad k_n\beta=2n+1\Rightarrow\exp(ik_n\beta)=-1$

$=\dfrac{-\exp\varepsilon_\nu\beta}{\exp\varepsilon_\nu\beta+1}\times(-e^{-\varepsilon_\nu\beta}-1)=1$. 原式 $=\dfrac{1}{ik_n-\varepsilon_\nu}$

$k_n=\omega\quad \omega\to i\omega+\eta\Rightarrow G=\dfrac{1}{\omega-\varepsilon_\nu+i\eta}$ 这与之前求Green function一致

松原频率是离散的，离散的级数无穷求和方可化为围道积分形式，数学上，如此定义 完全没问题

$$\mathcal{G}_{AB}(n)=\frac{1}{2}\int_{-\beta}^{\beta}d\tau\, e^{i\pi n\tau/\beta}\,\mathcal{G}_{AB}(\tau)$$

$$\mathcal{G}_{AB}(\tau)=\frac{1}{\beta}\sum_{-\infty}^{+\infty}e^{-i\pi n\tau/\beta}\,\mathcal{G}_{AB}(n)$$

后面几章中我们确要计算 $\mathcal{G}(\tau)$.，已知 $\mathcal{G}(\omega)$ 容易，Now 要求

① $S_1(\nu,\tau)=\frac{1}{\beta}\sum_{ik_n}\mathcal{G}_0(\nu,ik_n)e^{ik_n\tau}$ or 这样形式.

② $S_2(\nu,\tau)=\frac{1}{\beta}\sum_{ik_n}\mathcal{G}_0(\nu,ik_n)\,\mathcal{G}_0(\nu,ik_n+i\omega_n)e^{ik_n\tau}$

如何求和？(一个数学问题).

$$S_0^F(\tau) = \frac{1}{\beta}\sum_{ik_n} g_0(ik_n)\, e^{ik_n\tau}$$

$g_0 = g_{0_1}\cdot g_{0_2}\cdots$ 上文 G：$ik_n=\varepsilon_j$ 为奇点(简单) 令 $g=\prod_j \frac{1}{z-z_j}$

命. $n_F(z)e^{\tau z} = e^{\tau z}/e^{\beta z}+1$

围线 $\displaystyle\int_{C=Re^{i\theta}\ R\to\infty} \frac{dz}{2\pi i}\, n_F(z)\, g_0(z)\, e^{z\tau}$

又知 $\boxed{\tau>0,\ \beta>\tau}$, $\mathrm{Re}(\beta-\tau)$, $\mathrm{Re}(\tau)$.

项 $\dfrac{e^{\tau z}}{e^{\beta z}+1}$ ~~$z\to\infty$~~ $\propto$ $e^{(\tau-\beta)\mathrm{Re}z}$ $\mathrm{Re}z>0$；$e^{\tau\mathrm{Re}z}$ $\mathrm{Re}z<0$　$\left[e^{(R_1>0)\cdot Re^{i\theta}}+e^{(R_2<0)}\right]Re^{i\theta}\ J^{-1}$

∴ contour $\displaystyle\int_{C=Re^{i\theta}} dz(\to 0)\times e^{z\tau}$　对 $R\to\infty$ 总 $\to\infty$，(总)$^{-1}\to 0$

根据 Jordan 引理 for $a>0$. $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}dz=0$.

要求：τ：(右半区 τ)，$|f(z)|\le M_R$　$\lim_{R\to\infty} M_R=0$ 皆满足

总之：full contor $=0$.　(上面分析有瑕疵.

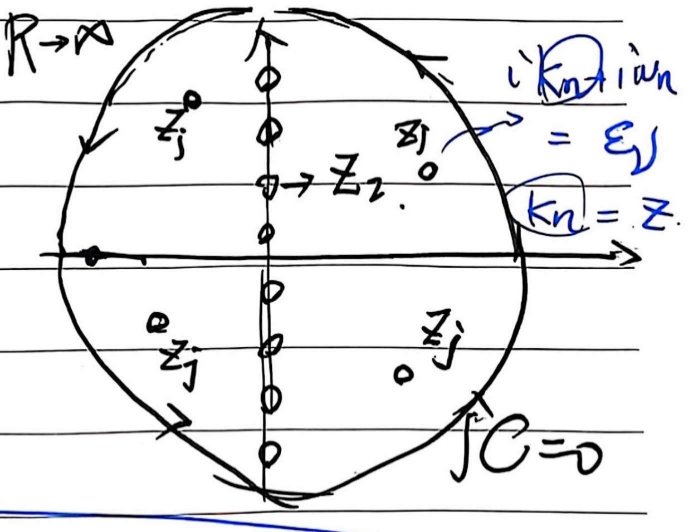
ie. $\displaystyle\int_{C=Re^{i\theta}\ R\to\infty} dz\, n_F(z)\, g_0(z)\, e^{z\tau} = 0$　↙这里应为 contor Jordan对的)

⇒ 围线积分可用留数求得

$n_F(z)g_0(z)e^{z\tau}$ Res 极点有 $g_0(z)$ 的 z_j

$n_F(z)$ 的：$\exp\frac{\beta z}{\beta}+1=0$

z_2 在虚轴上：$(2n+1)\pi/\beta$



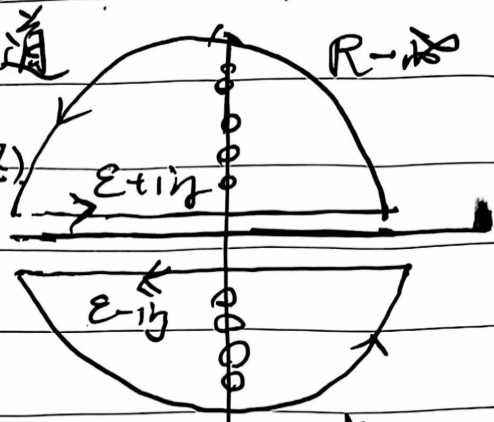
这正是松原频率！

ie. $\displaystyle\frac{1}{\beta}\sum_{i\omega_n} g_0(i\omega_n)e^{i\omega_n\tau} = -\sum_j \mathrm{Res}_{z=z_j}[g_0(z)]\, n_F(z_j)\, e^{z_j\tau}$　Fermi

$n_B(z_j)$　Boson.

$\mathrm{Res}_{z=(2n+1)\pi/\beta}\dfrac{p_0}{q_0} = \dfrac{p(z_0)}{q'(z_0)}$　$\dfrac{g_0 e^{i\omega_n\tau}}{\frac{1}{\beta}e^{z_0/\beta}\ldots}$

若存在相互作用 $g_0 \to g$ 不再~~是简单极~~的点，但若我们只考虑 G（没有 $G_a \times G_b \cdots$）i.e. $\sum g(ik_n)e^{ik_n\tau}$ 那么其极点将全部存在于实轴上。因（上一页 $z-E_n+E_{n'}$ 分母）作如图围道



围道积分为：$\int_{平行1+平行2} \frac{dz}{2\pi i} g(ik_n) e^{z\tau} n_F(z)$

$$= -\frac{1}{2\pi i}\int_R d\varepsilon\, n_F(\varepsilon)[g(\varepsilon+i\eta)-g(\varepsilon-i\eta)]e^{\varepsilon\tau}$$

when $\eta\to 0$ $g(\varepsilon+i\eta)-g(\varepsilon-i\eta) = g(\varepsilon+i\eta) - [g(\varepsilon+i\eta)]^*$

（仍利用 $\frac{\langle n|A|n'\rangle\langle n'|B|n\rangle}{z-E_n+E_{n'}}$ 式子 $z=\varepsilon+i\eta$ or $\varepsilon-i\eta$

代 $A=c_\gamma$ $B=c_\gamma^\dagger$

$= 2i\,\mathrm{Im}\, g^R(\varepsilon)$ 利用 G. g 对比性

G

$$\Rightarrow -\frac{2i}{2\pi i}\times \mathrm{Im}\,G^R(\gamma,\varepsilon) \rightsquigarrow A(\gamma,\varepsilon)$$

Therefore：上面留数和为：

$$\boxed{S_1(\gamma,\tau) = \frac{1}{\beta}\sum_{ik_n} g(\gamma,ik_n)e^{ik_n\tau} = \int_R \frac{d\varepsilon}{2\pi} n_F(\varepsilon)A(\gamma,\omega)e^{\varepsilon\tau}}$$

Equation of Motion：同理 对 $\hat H_0=\sum_{\gamma\gamma'} h_{0\gamma\gamma'} c_\gamma^\dagger c_{\gamma'}$

坐标/K空间 g 表示：$g(r\sigma\tau, r'\sigma'\tau') = -\langle T_\tau(\Psi_\sigma(r,\tau)\Psi^\dagger_{\sigma'}(r',\tau'))\rangle$

$g(k\sigma,k'\tau') = -\langle T_\tau(c_k(\tau)c^\dagger_{k'}(\tau'))\rangle$

$$-\partial_\tau g_0(r\tau r'\tau') - \int dr''\, h_0(r,r'')g_0(r''\tau,r'\tau') = \delta(\tau-\tau')\delta(r-r')$$

$$-\partial_\tau g_0(\gamma\tau,\gamma'\tau') - \sum h_{0\gamma\gamma''} g_0(\gamma''\tau,\gamma'\tau') = \delta(\tau-\tau')\delta_{\gamma\gamma'}$$

边值：$g_0(\tau) = \pm g_0(\tau+\beta)$

Wick 定理. n-particle Green's Function

$$\mathcal{G}_0^{(n)}(\nu_1\tau_1\cdots\nu_n\tau_n,\ \nu_1'\tau_1'\cdots\nu_n'\tau_n') = (-1)^n\langle T_\tau[c_{\nu_1}(\tau_1)\cdots c_{\nu_n}(\tau_n)c^\dagger_{\nu_n'}(\tau_n')\cdots c^\dagger_{\nu_1'}(\tau_1')]\rangle$$

$$= \sum_P (-1)^P\,\theta(\text{第一项}-\text{第二项})\cdots\theta(\text{第}n-1\text{项}-\text{第}n\text{项})\times\langle\text{一个}2n\text{排列}\rangle_0\,(-1)^n$$

产生.湮灭不可换 余下n中自排number = n!, $\overleftarrow{C_i^\dagger}\,|\,\overrightarrow{C_i}$ 位置对称 P为
为此排列交换算符次数, 原因就在于$\tau_1\cdots\tau_n$大小未知.

n-particle Green's Function Equation of Motion:

Wick 定理. Bruus. Chap. 11.

虚时 Green Function: $\mathcal{G}_0^{(n)}(\nu_1,\tau_1;\nu_2,\tau_2;\cdots\nu_n,\tau_n)$

$$= (-1)^n \langle T_\tau\, \hat{c}_{\nu_1}(\tau_1)\cdots \hat{c}_{\nu_n}(\tau_n)\,\hat{c}^\dagger_{\nu_n}(\tau_n')\cdots \hat{c}^\dagger_{\nu_1}(\tau_1')\rangle_0$$

where: $\hat{c}(\tau) = e^{\tau H_0}\, c\, e^{-\tau H_0}$

可以写作:

$$(-1)^n \sum_{p\in S_{2n}} (\pm1)^p\, \theta(\sigma_{p_1}-\sigma_{p_2})\cdots\theta(\sigma_{p_{n-1}}-\sigma_{p_n}).$$

$$\times\langle d_{p_1}(\sigma_{p_1})\cdots d_{p_{2n}}(\sigma_{p_{2n}})\rangle_0$$

where:

$$d_j(\sigma_j) = \begin{cases}\hat{c}_{\nu_j}(\tau_j) & j\in[1,n]\\ \hat{c}^\dagger_{\nu(2n+1-j)} & j\in[n+1,2n]\end{cases}$$

这就是一个排列 P eg: $C_1C_2C_3$ if $P=(3,1,2)$, 为 $C_3C_1C_2$ 如此排列前系数为 $\theta(C_3-C_1)\theta(C_1-C_2)$ $\theta(C_3-C_1)$ 为正或负取决于 $C_3(t_3)$ $C_1(t_1)$ t_3, t_1 之大小; $(\pm1)^P$ Fermion 每一个交换 c_ic_j · $\times(-1)$. Boson: $(\pm1)^P=1$

eg: $\langle \hat{C}_1(\tau_1=3)\, C_2(\tau_2=1)\, \cancel{C_3(\tau_3=5)\, C_3^\dagger(\tau_3'=0)}\, C_2^\dagger(\tau_2'=0)\, C_1$

$= (-1)^0\langle C_1C_2C_2^\dagger C_1^\dagger\rangle\cdot\theta(\underbrace{\tau_1-\tau_2}_{>0})\cdots + (-1)^1\langle C_1C_2C_1^\dagger C_2^\dagger\rangle\,\theta(\tau \ldots$

不交换

$+ (-1)^2\langle C_1C_1^\dagger C_2C_2^\dagger\rangle\,\theta(\tau_1-\tau_1') + (-1)^3\langle C_1^\dagger C_1C_2C_2^\dagger\rangle\,\theta(5-3)$

$+$ (24种; 含重复).

pf: Using Schodinger Equation:

$$-\frac{\partial}{\partial \tau_1} g_0^{n}(\dots) = \underbrace{-(-1)^n}_{\text{Equation of Motion.}} \left\langle T_\tau\left[I_1 H_0 \cdot c_1^{\dagger} I(\tau_1)\, \hat{c}_2(\tau_2) \cdots \hat{c}_n(\tau_n) \cdot \hat{c}_n^{\dagger}(\tau_n') \cdots c_{\nu_1}^{\dagger}(\tau_1')\right]\right\rangle_0$$

一开始 $c_n \cdot c_n^{+}$ 挨着，比如我们 $c_i(\tau_i)\, c_j^{+}(\tau_j')$ 挨着：

交换此两项.

$$\langle \cdots c_i(\tau_i)\, c_j^{+}(\tau_j') \cdots \rangle \underbrace{\theta(\tau_i - \tau_j')}_{T_\tau} \overset{\text{Fermion}}{\mp} \theta(\tau_j' - \tau_i) \langle \cdots c_j(\tau_j')\, c_i(\tau_i) \cdots \rangle$$

如果 Equation of Motion 求导恰为 τ_i 项，那么 $\frac{\partial}{\partial \tau_i} g^0$

$$\Rightarrow \frac{\partial}{\partial \tau_i}\theta(\tau_i - \tau_j') = \delta(\tau_i - \tau_j')$$ ① 对 c_i 求导 $[\cdots]\ (\theta - \theta)$ ② 对 θ 求导. $\delta \times \langle [c_i, c_j^{+}] \cdots \rangle$

$$= \delta(\tau_i - \tau_j) \left\langle \cdots\cdots [c_i(\tau_i), c_j^{+}(\tau_j')] \cdots\cdots \right\rangle$$

only: $i = j$ 非零. 故: 原式 $=$

$$\delta(\tau_i - \tau_j)\, \delta_{\nu_i \nu_j}\, \underbrace{(-1)^x}\, g_0^{(n-1)}(\nu\tau \cdots) \Rightarrow$$ 除去了 i, j 两项.

对于 x: Fermion: $\overset{自带}{(-1)} \cdot (-1)^n \cdot \underbrace{(-1)^{1-n} (-1)^{2n-i-j}}_{i,j \text{项交换至}(n, n+1)\text{项}} = (-1)^{j+i}$

Boson: $(-1)(-1)^n(-1)^{1-n}$ (Boson 括号正负) $= 1$

$$\because -\frac{\partial}{\partial \tau_i} g_0^{(n)} = g_{0i}^{(-1)} g_0^{(n)}$$ (上文 Equation of Motion 结论)

$$\therefore g_0^{(n)} = g_{0i}\left(-\frac{\partial}{\partial \tau_i} g_0^{(n)}\right)$$

$$= \sum_{j=1}^{n} (\pm)^{j+i} g_0(\nu_i \tau_i; \tau_j' \nu_j')\, g_0^{(n-1)}(\underbrace{\nu_1 \nu_1 \cdots \nu_n \tau_n \nu_1' \tau_1' \cdots \nu_n' \tau_n'}_{\text{去掉 } i, j \text{ 项}})$$

其中: $g_0(\nu_i \tau_i; \nu_j' \tau_j') \div$ (Single particle Green Function)

$\mathcal{G}$ " $G(t_1, t_2)$ "

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重复上述过程，看作 n个 g_0 的连乘：

$\therefore g^{(n)}$ 看作 $\begin{vmatrix} g^0 & & \\ & \ddots & \\ & & g^0 \end{vmatrix}$ 一个行列式组合.

Actually：

$$g_0^{(n)}(1\cdots n;\, 1'\cdots n') = \begin{vmatrix} g_0(1,1') & \cdots & g_0(1,n') \\ \vdots & & \\ g_0(n,1') & \cdots & g_0(n,n') \end{vmatrix} \quad i=(r_i, \tau_i)$$

That Results Wick's Theorem. $\langle \cdots \rangle = \langle\rangle\langle\rangle\langle\rangle \pm \langle\rangle\langle\rangle\langle \cdots + \langle$

Wick's Theorem so far says nothing but: How many minus sign should we multiple in many-body system? By consi

① Fermion anti-commute

② T_I: time ordering operator.

Example: Polarizability of free electrons.

上文，成功导出极化率用 ρ-ρ 关联函数之形式以及 Retard Function，方法是求 $\sum\langle[\rho,\rho]\rangle \rightsquigarrow \int_\Omega d\omega[\rho,\rho] \rightsquigarrow F[n_F(\varepsilon)]$

本文，再求之，应得出：Free-electrons 中：

$$\chi^R(q, t-t') = -i\theta(t-t')\frac{1}{V}\langle[\rho(q,t), \rho(-q,t')]\rangle$$

$$\rho(q,t) = \sum_{k\sigma} \underbrace{c_k^+ c_{k+q} e^{i(\varepsilon_k - \varepsilon_{k+q})t}}_{\text{Free-electron}}$$

$$\Rightarrow \chi_0^R(q, t-t') = \frac{1}{V}\sum_{k\sigma}\frac{n_F(\varepsilon_k) - n_F(\varepsilon_{k+q})}{\varepsilon_k - \varepsilon_{k+q} + \omega + i\eta}$$

此中：$-i\,\theta(t-t')\cdot[\rho(q,t),\rho(-q,t')]\frac{1}{V} = \langle T_\tau[\rho(q,\tau)\cdot\hat{\rho}(-q)]\rangle$

$= -\frac{1}{V}\langle T_\tau(\hat{C}^+_{k\sigma}(\tau)\hat{C}_{k+q\,\sigma}(\tau)\hat{C}^+_{k'\sigma'}(0)\hat{C}_{k'-q\,\sigma'}(0))\rangle_0$

Wick定理

$\Rightarrow \mathcal{G}_0^n = \sum \mathcal{G}_0\cdot\mathcal{G}_0^{(n-1)}\cdots \quad \mathcal{G}_0^{(n)}:\ \mathcal{G}_0(\underset{\nu_1'}{k}\cdot\underset{\nu_1}{k+q}\cdot\underset{\nu_2}{k'-q}\cdot\underset{\nu_2'}{k'})$

$\Rightarrow \mathcal{G}_0(\underset{\nu_1}{k+q}\cdot\underset{\nu_2'}{k'})\,\mathcal{G}_0(\nu_2\cdot\nu_1')\cdot(-1)^{\overbrace{i+j}}_{1+2}$

$+\mathcal{G}_0(\nu_1\cdot\nu_1')\,\mathcal{G}_0(\nu_2\cdot\nu_2')\cdot(-1)^{1+1} \Rightarrow$ 此项为：

$k'\neq k \Rightarrow \langle q\rangle_0\times\langle -q\rangle_0$

$\langle \underbrace{C_k^+C_{k+q}}_{\mathcal{G}_0}\rangle\langle\underbrace{C_{k'}^+C_{k'-q}}_{\mathcal{G}_0}\rangle_0$ $q\neq 0$时必为零　产生q与$k+q$内积为0.

余下：$-\frac{1}{V}\cdot(-1)\cdot\ \mathcal{G}_0(\underset{\tau}{k+q\cdot k'})\,\mathcal{G}_0(\underset{-\tau}{k-q\cdot k})$.

$= \frac{1}{V}\langle C_{k+q}^+C_k^{+}\rangle_0\langle C_{k'-q}^{+}C_{k'}^{+}\rangle_0$ only $k+q=k'$ 非零 $\sigma'=\sigma$

$= \frac{1}{V}\sum_{k\sigma}\mathcal{G}_0(k+q_\sigma,\tau)\,\mathcal{G}_0(k\sigma,-\tau)$

上文：无相互作用粒子(单)$\mathcal{G}_0$虚时 Green Function 为：

$$\mathcal{G}_0(k\sigma\cdot ik_n) = 1/(ik_n-\varepsilon_k)$$

Now：$\mathcal{G}_0((k+q)\sigma\cdot\tau) \xRightarrow{\text{Fourier}} \mathcal{G}(ik_n) = \int_0^\beta d\tau\, e^{i\,k_n\cdot\tau}\,\mathcal{G}(\tau)_{(k+q)}$　$k_n = \pi n/\beta$

$$\mathcal{G}_0(k+q\cdot ik_n+iq_n) = 1/(ik_n+iq_n-\varepsilon_{k+q})$$

$\Rightarrow \chi_0$ 同上文