

2026. Summer. CFT

conformal transformation. $f: \mathbb{Z} \rightarrow \mathbb{W}$.

$$g_{\mu\nu}(x) \xrightarrow{f} g'_{\mu\nu}(x') = \Delta(x) g_{\mu\nu}(x)$$

which preserves: local angles. $\angle g(x_1), g(x_2) \propto$

$\angle g'(x'_1), g'(x'_2)$ 切线夹角同

因此. $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$$x' = x + \epsilon(x) \text{ 满足 } (d-1) \partial^\mu \partial_\mu (\epsilon \cdot \epsilon) = 0$$

Therefore. transformation: $\epsilon(x) \leftarrow x$

$$\epsilon(x)_\mu = a_\mu + b_{\mu\nu} x_\nu + c_{\mu\nu\rho} x_\nu x_\rho$$

only four $\epsilon(x)_\mu$ satisfies: , form a group.

$$\epsilon_\mu = a_\mu$$

translation.

P_μ 动量.

$$\epsilon_\mu = \omega_{\mu\nu} x_\nu$$

rotation.

$J_{\mu\nu}$

$$\epsilon_\mu = \lambda x_\mu$$

dilatation.

D

$$\epsilon_\mu = b_\mu x^2 - 2x_\mu b_\nu x_\nu \text{ special } K_\mu$$

Field ϕ

$$\phi'(x) = \left| \frac{\partial x'}{\partial x} \right|^{\Delta/d} \phi(x) \text{ [primary fields]}$$

Δ : scaling dimension. $\left| \frac{\partial x'}{\partial x} \right|$ Jacobian

Example: $\phi(x) = \lambda^\Delta \phi(\lambda x)$ is a primary field

It's ϕ with conformal transformation invariance

Operators O with Δ_i

under primary fields, $O(x_i)$ should follow correlation of power decay.

$$\langle O_i(x_1) O_j(x_2) \rangle = \delta_{\Delta_i \Delta_j} \frac{1}{|x_1 - x_2|^{2\Delta_i}}$$

$$\langle O_i(x_1) \rangle = \delta_{\Delta_i, 0}$$

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{f_{ijk}}{|x_{12}|^{\Delta_i + \Delta_j - \Delta_k} |x_{23}|^{\Delta_j + \Delta_k - \Delta_i} |x_{13}|^{\Delta_k + \Delta_i - \Delta_j}}$$

we aim to solve CFT by, we want to know conformal data. (Δ_i, f_{ijk}) operation product expansion.

In RG flow:

$$\frac{dg_k}{d\ell} = (d - \Delta_k) g_k - \sum_{ij} g_{ij} f_{ijk} g_i g_j$$

Example: If $G(x_1, x_2) = \langle \phi(x_1) \phi(x_2) \rangle$ conformal.

we apply translation and dilatation.

$$x \rightarrow x + a$$

$$x \rightarrow \lambda x$$

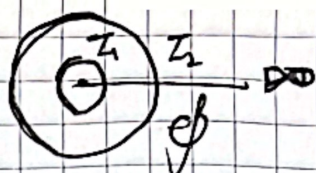
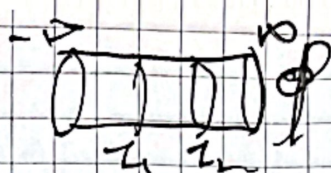
$$G(x_1, x_2) = \frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \quad C = F(1)$$

we apply special conformal $\textcircled{4}$: $\Delta_1 = \Delta_2$.

If conformal, only Δ is unknown.

$\uparrow$
under conformal Group: [4 elements]

In a cylinder



$$\phi_{\text{cyl}}(z, \Omega) = \Lambda(r, \Omega)^{\Delta/2} \phi_{\text{plane}}(z, \Omega)$$

power

$$\phi_{\text{cyl}}(z, \Omega) = \Lambda(r, \Omega)^{\Delta/2} \langle \phi_{\text{plane}}(z) \phi_{\text{plane}}(0) \rangle$$

$$\langle \phi_{\text{cyl}}(z, \Omega) \phi_{\text{cyl}}(0) \rangle$$

$$R^{-2} e^{r\Delta/R} R^{-2} \frac{1}{|r-1|^{2\Delta}}$$

$$e^{-\Delta/R}$$

Therefore: conformal transform

to a cylinder, the power law becomes exponential decay

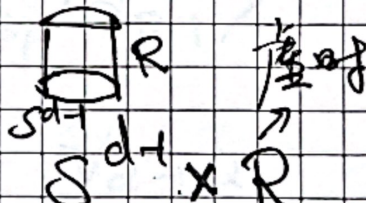
$$\Delta E \sim \frac{1}{3} \sim \frac{\Delta}{R} \text{ [Gap]}$$

(Lattice length)

State operator correspondence

Gap $\sim$ scaling dimension.

柱面上, 认为 x 为时间. 等价于虚时演化. Gap 有关指数. 柱面上 (PBC) 有离散谱



plane: R^d

cylinder

临界 (CFT)

临界才有 scaling law

$$H = -J \sum \sigma_i \sigma_{i+1}$$

$$H = \sqrt{\frac{J}{2}} \sum \sigma_n^x \sigma_{n+1}^z + h \sum \sigma_n^x$$

2-D classical Ising

1-D Ising - 横场 (临界)

$$\langle \sigma(w) \sigma(r) \rangle = \left(\frac{2\pi}{\Delta} \right)^r$$

$$\Delta = \ln(2\pi/\Delta)$$

($\Delta/2$ odd)

$$\Delta = 1/2$$

$\downarrow$ JW-transform $h=1$

$\downarrow$ 基态, 得到 Gap

$\downarrow$ 得到 scale dimension

CFT

$$(1+1)D: u, (1,2) \text{ transformation: } \epsilon_\mu \bar{z} \rightarrow f(z) = z + \epsilon(z)$$

$$\left(\frac{\partial \epsilon_\mu}{\partial x_\mu} + \frac{\partial \epsilon_\nu}{\partial x_\nu} \right) = f(x) g_{\mu\nu}(x) \quad f(x) = \frac{2}{2d} \partial_\mu \epsilon^\mu$$

$$\Rightarrow \partial_1 \epsilon_1 = \partial_2 \epsilon_2 \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$$

(1+1)D analytical function is infinite, there is infinite local conformal transformations, only valid on 2D. $\mathbb{C}$ plane, Local CFT has singularity. ϵ 生成元: $\ln$, expand below Laurent

$$e^{c \ln + \bar{c} \bar{\ln}} \psi(z, \bar{z}) = \psi(z - \epsilon(z), \bar{z} - \epsilon(\bar{z}))$$

$$\ln = -z^{n+1} \partial_z; \quad \bar{\ln} = -\bar{z}^{n+1} \partial_{\bar{z}}$$

$$[\ln, \ln] = (m-n) \ln_{m+n} \quad [\bar{\ln}, \bar{\ln}] = (m-n) \bar{\ln}_{m+n}$$

$$[\ln, \bar{\ln}] = 0 \quad \text{Witt algebra}$$

Any conformal transformation $V(z)$ is:

$$V(z) = \sum_n a_n \ln$$

we require analytical, i.e.

$$z \rightarrow 0, \text{ finite}; \quad z \rightarrow \infty, \text{ finite} \Rightarrow [n \leq -1, n \leq 1] \quad n=0 \pm 1$$

$$l_{-1} = \partial_z = \frac{1}{2} (\partial_0 - i \partial_1) \quad \text{Global CFT}$$

Four global conformal transformation:

translation: $l_1 + \bar{l}_{-1}, i(l_1 - \bar{l}_{-1})$

rotation: $i(l_0 - \bar{l}_0)$

dilatation: $l_0 + \bar{l}_0$

special: $l_1 + \bar{l}_1, i(l_1 - \bar{l}_1)$

Any analytical in 2D $\Rightarrow$ Global CFT $\rightarrow$ Local (No singularity)

$$[L_n, L_m] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n, 0}$$

c : central charge

Quantum anomaly

$L_n \xrightarrow{\text{quantization}} L_n$

plane $\rightarrow$ cylinder $\mathbb{R}^N \rightarrow S^{N-1} \times \mathbb{R}$

$D(\psi, \bar{\psi}) \rightarrow H(\psi, \bar{\psi}) = \frac{c}{6} \frac{\pi}{L}$ Degrees of freedom

Their 2 零点能贡献能量

能动张量: $L[\phi, \partial_\mu \phi]$ satisfies:

$$\partial_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi} \partial^\mu \phi - L g^{\mu\nu} \right] \epsilon_\nu = 0$$

if conformal: $T^{\mu\nu}$

$$\delta S = \frac{1}{d} \int d^d x T^{\mu\nu} g_{\mu\nu} \partial_\rho \epsilon^\rho = \frac{1}{d} \int d^d x T^{\mu}_{\mu} \partial_\rho \epsilon^\rho = 0$$

in 2D, T^{μ}_{μ} traceless $\Rightarrow T_{z\bar{z}} = 0, T_{\bar{z}z} = 0$

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) + \dots$$

$$T(z) = \sum_n z^{-n-2} L_n$$

(1+1) free boson. ϕ $\Delta=0$ $\partial_z \phi$ $\Delta=1$

$$T(z) = \partial_z \phi \partial_z \phi \quad \text{需要正规序} \quad \boxed{\text{量子化}}$$

$\Downarrow$ Quantum

$$T(z) = -2\pi : \partial_z \phi \partial_z \phi : = -2\pi \lim_{w \rightarrow z} \left[\partial_z \phi(z) \partial_w \phi(w) - \langle \partial_z \phi(z) \partial_w \phi(w) \rangle \right]$$

$$T(z) T(w) = 4\pi : : \quad \quad \quad - \langle \partial_z \phi(z) \partial_w \phi(w) \rangle$$

$$\text{Wick} \quad \frac{1/2}{(z-w)^4} = 4\pi \frac{\partial_z \phi \partial_z \phi}{(z-w)^2}$$

坐标变换时 Noether 荷 $Q = \int d^{d+1} x \xi_\mu(x) T^{\mu\nu} \rightarrow T$

eg: $C = 1/2 ; 7/10$ $C = 1 - \frac{6}{m(m+1)}$ $m = 3, 4, \dots$
 $\uparrow$
Ising

Quantum Anomaly.

$$(0+1)D. S_{\psi} = \int \bar{\psi} \overset{\text{Grassmann}}{\partial_z} \psi dz.$$

charge conjugation system. 电荷反演对称性.

classical $S_A(r, \bar{r}) = S_{-A}(\bar{r}, r)$

= 交换电荷, 是否破坏物理规律. r : electron $\bar{r}$: 正电子 $-e$.

$$S = S' = \int \psi \partial_z \bar{\psi} dz = \int dz (\partial_z \psi) \bar{\psi} = \int dz \bar{\psi} \partial_z \psi$$

Global $U(1)$: $\psi'(z) = e^{i\alpha} \psi(z)$

$$\bar{\psi}'(z) = e^{-i\alpha} \bar{\psi}(z)$$

判断对称性. 在如此变换下, S 是否有. $S[\psi, \bar{\psi}'] = S[\psi, \bar{\psi}]$

Local $U(1)$: $\psi'(z) = e^{i\alpha(z)} \psi(z)$

$$\bar{\psi}'(z) = e^{-i\alpha(z)} \bar{\psi}(z).$$

Therefore: $S = \int \bar{\psi} \partial_z \psi dz$ No Local $U(1)$

Another. $S_A[\psi, \bar{\psi}] = \int dz \bar{\psi}(z) [\partial_z + iA(z)] \psi(z).$

Gauge. $A'_z = A(z) - \partial_z \alpha(z)$ satisfies.

$$S_{A'}[\psi', \bar{\psi}'] = S_A[\psi, \bar{\psi}] \quad U(1).$$

$$S_{-A}[\bar{\psi}, \psi] = S_A[\psi, \bar{\psi}]$$

↑
charge conjugation

Quantum:

$$-\int d\tau \bar{\psi} \partial_\tau \psi$$

$$Z = \int D[\psi, \bar{\psi}] \exp(-S[\psi, \bar{\psi}])$$

with APBC. (从 $\text{Tr} e^{-\beta H}$ 变换至路径积分自带头尾边界)

$$\psi(z+\beta) = -\psi(z) \quad \bar{\psi}(z+\beta) = -\bar{\psi}(z)$$

Fourier: $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n \exp[-\frac{i2\pi}{\beta}(n+\frac{1}{2})z]$ 满足 APBC.

$$\bar{\psi}(z) = \sum_{n \in \mathbb{Z}} \bar{\psi}_n \exp(i\frac{2\pi}{\beta}(n+\frac{1}{2})z)$$

$$S[\psi, \bar{\psi}] = \exp(\sum_n i2\pi(n+\frac{1}{2}) \bar{\psi}_n \psi_n)$$

$$\int D[\psi, \bar{\psi}] = \int \prod_{n \in \mathbb{Z}} d\psi_n d\bar{\psi}_n \quad \text{截断 } \frac{N_2}{-N_1}$$

$$Z = \prod_{n \in \mathbb{Z}} C_n = \prod_{n \in \mathbb{Z}} 2\pi i (n+\frac{1}{2})$$

charge conjugation: $\begin{cases} \psi \rightarrow \bar{\psi} \\ \bar{\psi} \rightarrow \psi \end{cases} (z) \Rightarrow \begin{cases} \psi_n \rightarrow \bar{\psi}_{-n-1} \\ \bar{\psi}_n \rightarrow \psi_{-n-1} \end{cases}$

测度: $\frac{N_2}{-N_1} \int d\psi'_n d\bar{\psi}'_n = \frac{N_2}{-N_1} \int d\bar{\psi}_{-n-1} d\psi_{-n-1}$

... $d\bar{\psi}_n \dots d\bar{\psi}_{-n-1} d\psi_{-n-1}$
 换回来, 要换 $(-1)^{N_1+N_2-1}$

$$= \frac{N_2}{-N_1} (-1)^{N_1+N_2-1} [d\psi_{-n-1} d\bar{\psi}_{-n-1}]$$

" N_2 " $\frac{N_1-1}{-N_1} \cdot (-1)^{N_1+N_1} [d\psi_n d\bar{\psi}_n]$
 $\tilde{n} = -N_2-1 = -N_1$ set = " $N_2 = N_1-1$ " final:

$\mathcal{H} = \dim \mathcal{H}$
 $\mathcal{H} = \pi \psi \partial \psi - L$
 $= 0$
 与 β 无关:
 基态仅 2 个态
 简并
 能量为零无激发

测度满足 C.C. 对称 $\Rightarrow K = n + \frac{1}{2}$

$$Z = \prod_{K=-N_1-\frac{1}{2}}^{N_1-\frac{1}{2}} 2\pi i \cdot K = 2 = \text{Tr} e^{-\beta H}$$

Zeta-function

Now. $A \neq 0$

$$Z[A] = \int D[\psi, \bar{\psi}] \exp. \left[- \int \bar{\psi} (\partial_z + i A(z)) \psi dz \right]$$

$$\psi'(z) = e^{i\alpha(z)} \psi(z) \quad A'(z) = A(z) - \partial_z \alpha(z)$$

small gauge transformation $\alpha(0) = \alpha(\beta)$ ✓ use + h.c.

large gauge transformation $\alpha(0) + 2\pi = \alpha(\beta)$. $\alpha(z) = \frac{2\pi}{\beta} z$

C.C. $Z[A] = Z[A]$ 测度自动满足, $S_A = S_{-A}$ 满足 ✓
 Local $U(1)$. $Z[A] = Z[A - \partial\alpha]$ $\frac{2\pi}{\beta}$

$$\text{Let } A(z) = A \quad S_A[\psi, \bar{\psi}] = \exp \sum_{n \in \mathbb{Z}} \left[2\pi i \left(n + \frac{1}{2} - \frac{A\beta}{2\pi} \right) \bar{\psi}_n \psi_n \right]$$

测度自动满足.

$$Z[A] = \prod_{n=-N_1}^{N_1-1} 2\pi i \left(n + \frac{1}{2} - \frac{A\beta}{2\pi} \right)$$

$$Z[A - \frac{2\pi}{\beta}] = \prod_{n=-N_1}^{N_1-1} 2\pi i \left(n + \frac{1}{2} + 1 - \frac{A\beta}{2\pi} \right)$$

$$Z_{-N_1} \approx Z_{N_1-1} \times (-1)$$

$$Z[A - \frac{2\pi}{\beta}] = - Z[A] \cdot \frac{N_1 + \frac{1}{2} - \frac{A\beta}{2\pi}}{N_1 - \frac{1}{2} - \frac{A\beta}{2\pi}} \xrightarrow{N_1 \rightarrow \infty} - Z[A]$$

令 $\alpha(0) + 2\pi N = \alpha(\beta)$. 经过多少次 spectral flow.

即 $Z[A - \frac{2\pi}{\beta}] = (-1)^N Z[A]$ 谱不用 $U(1)$ 规范变换而变. 是集体平移.

Therefore, it's CC preserving gauge transformation. (regularization).

To preserve $U(1)$, we should

choose 测度 to cancel S_A 能谱平移

而 $N \rightarrow \infty$ 全部乘完. 自然不平移. 但测度无法定义.

$$U(1): \prod_{n=-\infty}^{\infty} \lambda_n = \exp \left(\sum_n \ln \lambda_n \right) \text{ 发散}$$

额外乘以: $\lim_{S \rightarrow 0} \exp \left(\sum_n e^{-S|\lambda_n|} \ln \lambda_n \right)$ 避免发散

不满足 C.C. 不对称能谱.

或: 目标 Directly find: $Z_{ucj}[A] = Z_{ucj}(A - \frac{2\pi}{\beta})$

Def: $Z_{ucj}[A] = Z_{c.c}[A] \cdot \exp\left(\frac{iA\beta}{2}\right)$

可行!

Mixed anomaly

A 含时: $\exp\left[\frac{i}{2} \oint A(z) dz\right]$

是 $U(1)$ 和 C.C 正规化项之差.

一个成立, 另一个无效不成立.

Majorana. $\mathcal{L}(\psi) = \psi \partial_z \psi$ Only One

$$Z = \int \prod (d\psi_n) \cdot \exp \left[2\pi i \psi_{-n} \left(n + \frac{1}{2}\right) \psi_n \right]$$

$$= \int \prod_{n=-\infty}^{\infty} (d\psi_n d\psi_{-n}) \exp \left[2\pi i \psi_{-n} \left(n + \frac{1}{2}\right) \psi_n \right]$$

$$= \left[\prod_{n=1}^{\infty} 2\pi i \left(n + \frac{1}{2}\right) \right] = \sqrt{2} \quad \text{Gravitational Anomaly.}$$

$$\dim H = \sqrt{2}$$

Fermion. $\exists \bar{\psi}$, with ψ . $\mathbb{Z}_2^F$ symmetry. $\psi' = -\psi$

$$\psi = \sum_{n \in \mathbb{Z}} \psi_n \exp\left(-i \frac{2\pi}{\beta} n\right)$$

$$= \sum_{n \in \mathbb{Z}} \psi_n \psi_{-n} 2\pi i n : \mathbb{Z}_2^F$$

Z-测度: $\int d\psi_0 \times \dots \int d\psi_1 d\psi_{-1} \times \dots \int d\psi_n d\psi_{-n}$

$$D(\psi) = -D(\psi) \quad \text{零能模}$$

$$= -d\psi_0$$