

Fitting Luttinger Parameter

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Review



1-D spinless fermion at $\pm k_F$, expand low energy excitation to left/right moving field, further expand to two bosonic field.

$$\Psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)$$

$$\psi_r(x) = \frac{\eta_r}{\sqrt{2\pi a}} :e^{-i[r\phi(x) - \theta(x)]}:$$

$$\rho(x) = \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L + e^{-2ik_F x} \psi_R^\dagger \psi_L + e^{2ik_F x} \psi_L^\dagger \psi_R$$

$$= \rho_0 + \underbrace{\frac{1}{\pi} \partial_x \phi(x)}_{\text{smooth part}} + \underbrace{A_1 \cos[2k_F x - 2\phi(x) + \delta_1]}_{\text{cross term}}$$

Background density by normal ordering, i.e. k_F/π

where, $\frac{\eta_R^\dagger \eta_L^\dagger}{2\pi a} = A_1$, δ_1 usually 0

If other terms. i.e. unklapp. U . Δ is irrelevant. low energy hamiltonian of Luttinger Liquid, following the sine-Gordon universality can be written as:

$$\mathcal{H}_{LL} = \frac{u}{2\pi} \int dx \left[K (\partial_x \theta)^2 + \frac{1}{K} (\partial_x \phi)^2 \right]$$

$$S[\phi] = \frac{1}{2\pi K} \int dx d\tau \left[\frac{1}{u} (\partial_\tau \phi)^2 + u (\partial_x \phi)^2 \right]$$

Define: $G(x, y) = \langle \phi(x) \phi(y) \rangle = G(x-y)$

$G(q, i\omega)$ is a Gauss Integral of $S[\phi]$, we have

$$G(q, i\omega) = \frac{\pi K u}{\omega^2 + q^2 u^2}$$

$$G(\omega) - G(r) = \frac{K}{2} \ln \frac{|r|}{a}$$

$$= \langle [\phi(r) - \phi(r_0)]^2 \rangle \cdot \frac{1}{2}$$

$$\langle e^{i\lambda x} \rangle = e^{-\frac{\lambda^2}{2} \langle x^2 \rangle}$$

$$\Rightarrow \langle e^{2i\phi(r)} e^{-2i\phi(\omega)} \rangle = \left(\frac{a}{|r|} \right)^{2K}$$

Density channel



$$C_{\rho\rho}(x) = \langle \rho(x) \rho(0) \rangle - \langle \rho(x) \rangle \langle \rho(0) \rangle$$

$$\Rightarrow \frac{1}{\pi^2} \partial_x \partial_y \langle \phi(x) \phi(y) \rangle \Big|_{y=0}$$

$$= \frac{1}{\pi^2} - \frac{d^2 G \phi(r)}{dr^2} = -\frac{K}{2\pi^2 x^2}$$

$$\Rightarrow A_1^2 \langle \cos[2k_F x - 2\phi(x)] \rangle \langle \cos[2\phi(0)] \rangle$$

$$= A_1^2 \cdot \text{Re} \left\{ e^{2ik_F x} \langle e^{-2i[\phi(x) - \phi(0)]} \rangle \right\}$$

$$= \frac{A_1^2 2^{2K}}{2} \frac{\cos(2k_F x)}{|x|^{2K}}$$

$$C_{\rho\rho}(x) = -\frac{K}{2\pi^2 x^2} + B_1 \frac{\cos(2k_F x)}{|x|^{2K}}$$

We also can calculate $\langle \rho(x) \rangle$ at OBC: $e^{-2\langle \phi(x)^2 \rangle}$

$$\langle \rho(x) \rangle = \rho_0 + A_1 \cos(2k_F x - 2\phi_b) \langle e^{-2i\phi(x)} \rangle$$

Here, we choose a constant background ϕ_b that $\phi(x) = \phi_b + \psi(x)$ and $\psi(0) = \psi(L) = 0$ for OBC.

we can choose

$$\psi(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi}{L} x\right)$$

$$S = \frac{2}{\pi} \frac{1}{2\pi K} \int dx d\tau \left\{ \frac{1}{u} [\partial_\tau \phi(x, \tau)]^2 + u [\partial_x \phi(x, \tau)]^2 \right\}$$

Using: $\int_0^L \sin(k_n x) \sin(k_m x) dx = \frac{L}{2} \delta_{m,n}$

$$S = \frac{1}{2\pi K} \sum_{n=1}^{\infty} \int d\tau \left(\frac{\partial_\tau q_n^2}{u} + 4k_n^2 q_n^2 \right)$$

$$= \frac{1}{2} \sum_n \int \frac{d\omega}{2\pi} \cdot \frac{\omega^2 + u^2 k_n^2}{\pi K u} q_n(\omega) q_n(-\omega)$$

Also is the Gauss Integral. we have.

$$\langle q_n q_m \rangle = \delta_{m,n} \frac{\pi K}{2k_n}$$

$$\langle \psi(x)^2 \rangle = \frac{K}{2} \ln \left[\frac{2L}{\pi a} \sin \frac{\pi x}{L} \right]$$

$$\langle \rho(x) \rangle = \rho_0 + A_1 \cos(2k_F x - 2\phi_b) \cdot \left[\frac{\pi a}{2L \sin(\pi x/L)} \right]^K$$

Let: $A = A_1 2^K$ $\delta = -2\phi_b$

$$\langle \rho(x) \rangle = \rho_0 + A \frac{\cos(2k_F x + \delta)}{\left[\frac{2L}{\pi} \cdot \sin \frac{\pi x}{L} \right]^K}$$

Bond channel



$$X = (j + \frac{1}{2})a \quad x_- = X - \frac{a}{2} \quad x_+ = X + \frac{a}{2}$$

$$c_j^\dagger c_{j+1} = a \left[e^{ik_F a} \psi_R^\dagger(x_-) \psi_R(x_+) + e^{-ik_F a} \psi_L^\dagger(x_-) \psi_L(x_+) \right. \\ \left. + e^{-2ik_F X} \psi_R^\dagger(x_-) \psi_L(x_+) + e^{2ik_F X} \psi_L^\dagger(x_-) \psi_R(x_+) \right]$$

$$\psi_r^\dagger(x_+) = \psi_r^\dagger(X) + \frac{a}{2} \partial_X \psi_r(X) + O(a^2)$$

$$\psi_R^\dagger(x_+) = \psi_R^\dagger + \frac{a}{2} \dot{\psi}_R^\dagger + \frac{a^2}{8} \ddot{\psi}_R^\dagger + \dots$$

$$\psi_R^\dagger(x_-) = \psi_R^\dagger - \frac{a}{2} \dot{\psi}_R^\dagger + \frac{a^2}{8} \ddot{\psi}_R^\dagger - \dots$$

$$\psi_L(x_+) = \psi_L + \frac{a}{2} \dot{\psi}_L + \frac{a^2}{8} \ddot{\psi}_L + \dots$$

$$\psi_L(x_-) = \psi_L - \frac{a}{2} \dot{\psi}_L + \frac{a^2}{8} \ddot{\psi}_L - \dots$$

$$c_j^\dagger c_{j+1} + h.c. \Rightarrow \left[\psi_R^\dagger(x_-) \psi_L(x_+) + \psi_R^\dagger(x_+) \psi_L(x_-) \right] e^{-2ik_F X}$$

$$= 2\psi_R^\dagger \psi_L + \frac{a^2}{4} \left[\psi_R^\dagger \ddot{\psi}_L + \ddot{\psi}_R^\dagger \psi_L - 2\dot{\psi}_R^\dagger \dot{\psi}_L \right]$$

$$\Rightarrow e^{ik} \psi_R^\dagger(x^-) \psi_R^\dagger(x^+) + e^{-ik} \psi_R^\dagger(x^+) \psi_R^\dagger(x^-)$$

$$= (e^{ik} + e^{-ik}) \psi_R^\dagger \psi_R + \frac{a}{2} (e^{ik} - e^{-ik}) (\dot{\psi}_R^\dagger \dot{\psi}_R - \dot{\psi}_R^\dagger \dot{\psi}_R)$$

$$= 2\cos(k_F a) \psi_R^\dagger \psi_R + ia \sin(k_F a) (\dot{\psi}_R^\dagger \dot{\psi}_R - \dot{\psi}_R^\dagger \dot{\psi}_R)$$

$$c_j^\dagger c_{j+1} + h.c. = a \psi_R^\dagger \psi_L + h.c. + 2\cos(k_F a) (\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)$$

$$+ ia^2 \sin(k_F a) (\dot{\psi}_R^\dagger \dot{\psi}_R - \dot{\psi}_R^\dagger \dot{\psi}_R - \dot{\psi}_L^\dagger \dot{\psi}_L + \dot{\psi}_L^\dagger \dot{\psi}_L)$$

$$\Rightarrow B_0 = \frac{2a\cos k_F a}{\pi} \partial_X \phi - \frac{a^2}{\pi} \sin(k_F a) [(\partial_X \phi)^2 + (\partial_X \theta)^2]$$

$$\langle c_j^\dagger c_{j+1} + h.c. \rangle_{\text{Sn}} = B_0 - 0 = \frac{a^2}{\pi} \sin(k_F a) [\langle (\partial_X \phi)^2 \rangle + \langle (\partial_X \theta)^2 \rangle]$$

$$\langle c_j^\dagger c_{j+1} + h.c. \rangle_{\text{osc}} = \langle a \psi_R^\dagger \psi_L \rangle = \frac{A_0 \cos(k_F x + \delta)}{[\frac{2L}{\pi} \sin \frac{\pi x}{L}]^k}$$

similarly, for OBC, we have:

$$\psi(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} g_n \sin(k_n X)$$

$$\pi(X) = \frac{1}{\pi} \partial_X \theta = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} P_n \sin(k_n X)$$

$\mathcal{H} = \sum_n \left(\frac{4\pi K}{2} P_n^2 + \frac{u k_n^2}{2\pi K} g_n^2 \right)$ so (P, g) can be expressed as harmonic oscillation operators: $(b_n^\dagger, b_n)$.

$$\psi(x) = \sum_n \sqrt{\frac{\pi K}{L k_n}} \sin(k_n X) (b_n + b_n^\dagger)$$

$$\theta(x) = \theta_0 + \frac{i}{\sqrt{K}} \sum_{n=1}^{\infty} \frac{\cos(k_n X)}{\sqrt{n}} (b_n - b_n^\dagger)$$

at G.S. we know b operators follow:

$$\langle b_n \rangle = 0, \quad \langle b_n b_m^\dagger \rangle = \delta_{nm}, \quad \langle b_n^\dagger b_m \rangle = 0.$$

Pair channel ?



$$\text{Therefore: } \langle (\partial_x \phi)^2 \rangle = \frac{\kappa \pi^2}{L^2} \sum_{n=1}^{\infty} n e^{-n\pi a/L} \cos^2 \frac{n\pi x}{L}$$

$$\langle (\partial_x \theta)^2 \rangle = \frac{\pi^2}{\kappa L^2} \sum_{n=1}^{\infty} n e^{-n\pi a/L} \sin^2 \frac{n\pi x}{L}$$

$$\Rightarrow \frac{a^2 \sin(k_F a)}{2\pi} (\kappa - \kappa^{-1}) \cdot \frac{1}{\left(\frac{2\pi}{L} \sin \frac{\pi x}{L}\right)^2}$$

therefore .

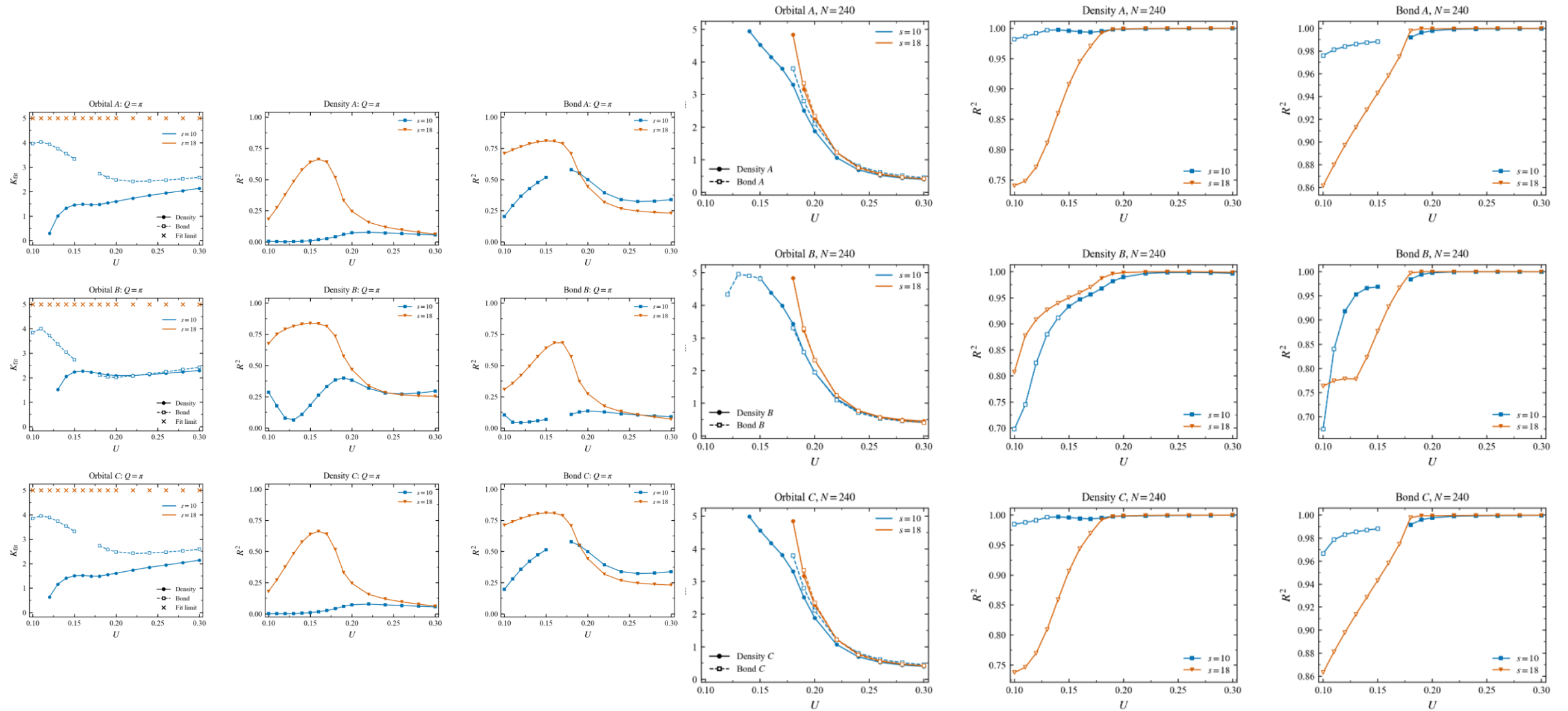
$$\langle c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \rangle = B_0 + \frac{C_B(k)}{(d_B X)^2} + \frac{A_B \cos(k_F X + \delta)}{(d_B X) K}$$

For pairing channel, if we calculate $\langle c_j c_{j+1} \rangle$,
for the original sine-Gordon universality, it's a particle number
conserving system. it's zero. ie: $\langle c_j c_{j+1} \rangle$ calculates

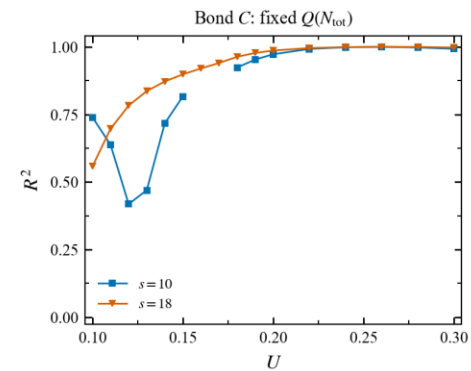
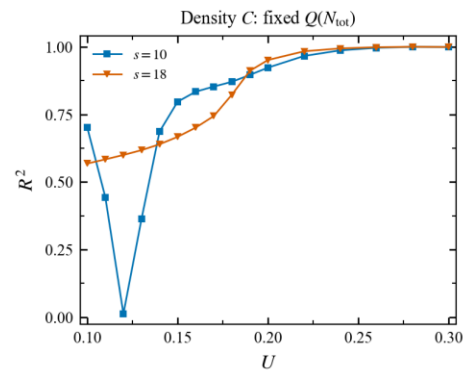
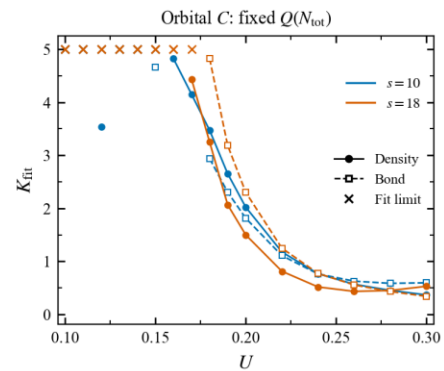
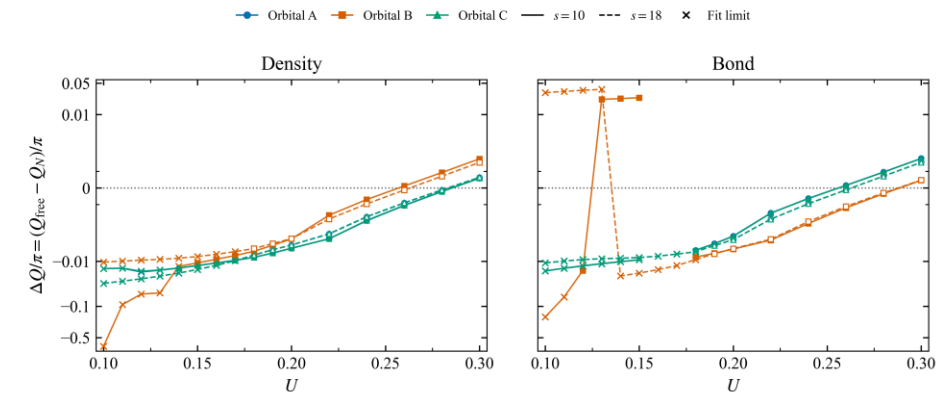
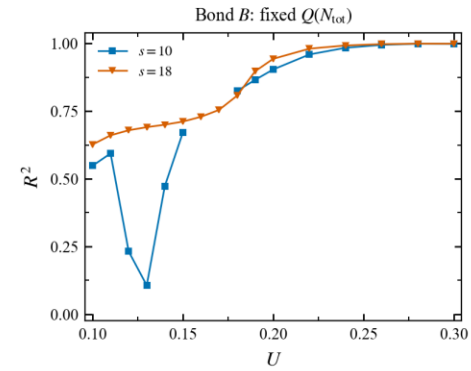
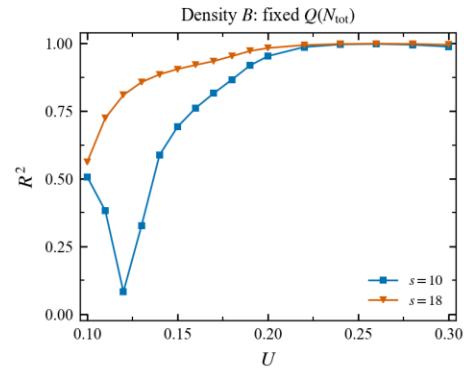
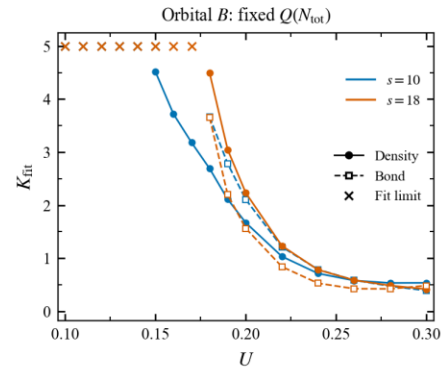
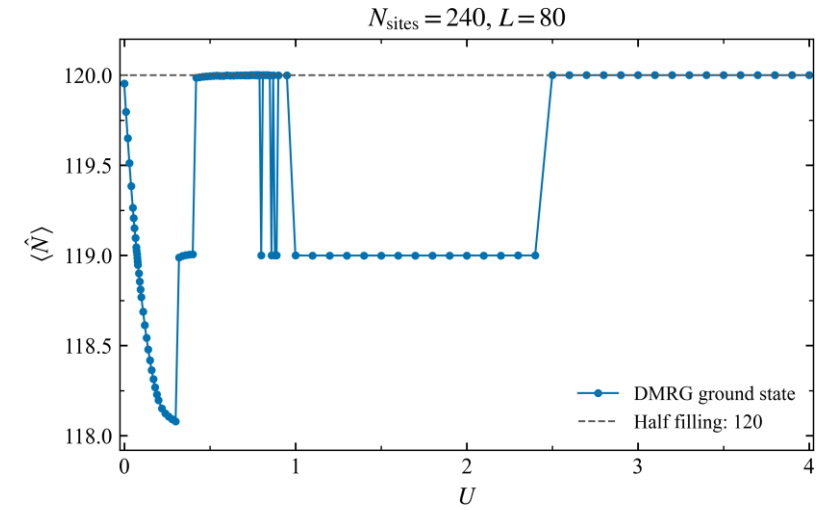
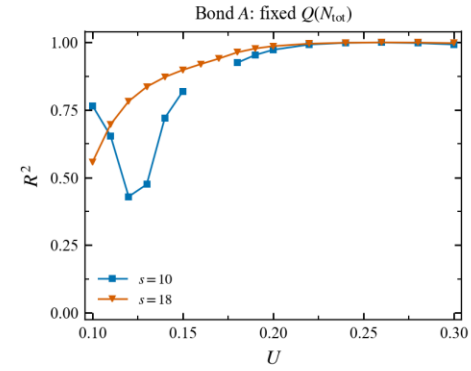
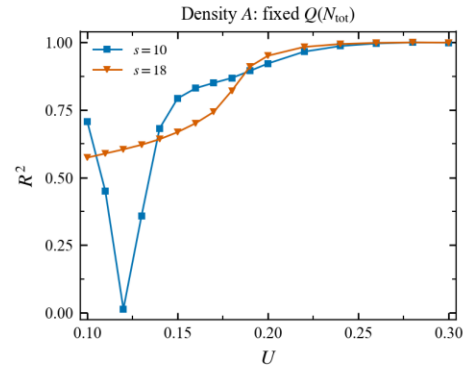
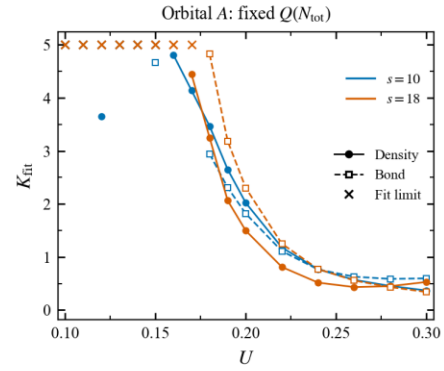
$e^{-\langle \theta(x) \rangle}$ and $\theta(x)$ don't have a fixed phase. so $\langle e^{i\theta} \rangle = 0$.

the IPDW is driven by Δ , a relevant operator to this hamiltonian
it's a perturbation to original hamiltonian. these sayings above
don't work to pairing channel.

Fit try



Fit try



Fit try

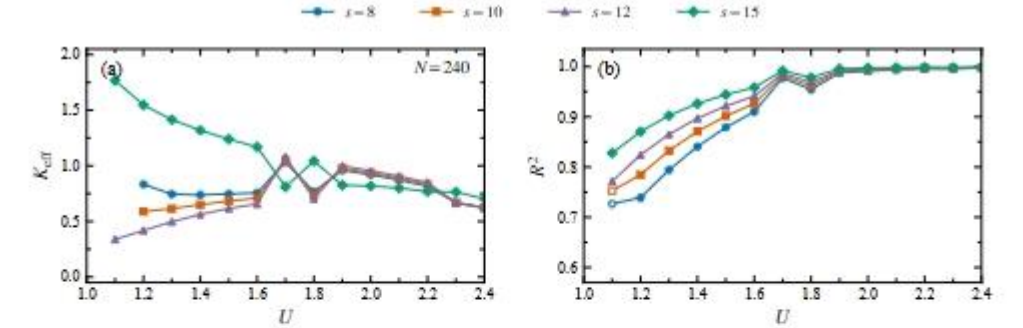
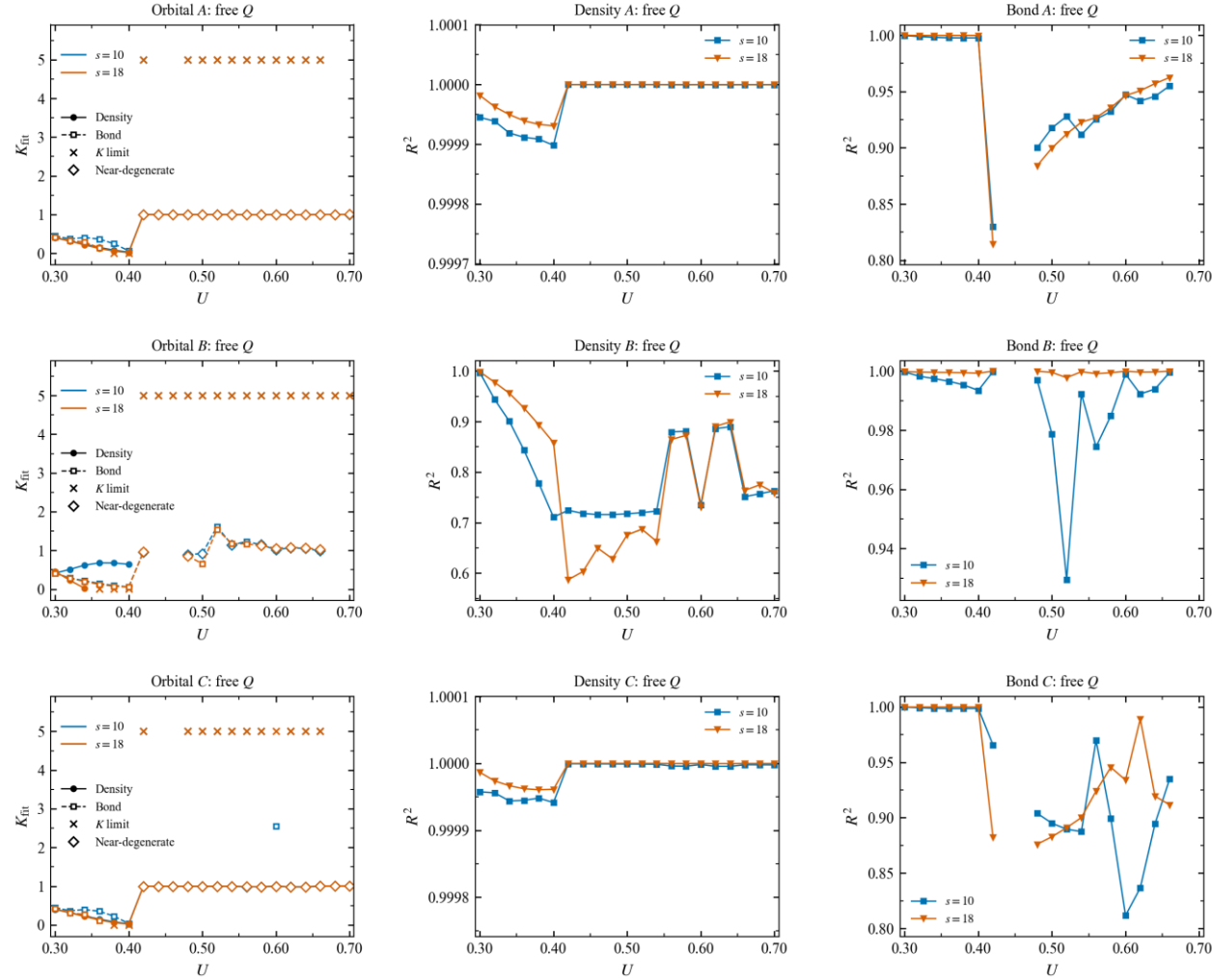


FIG. 8. Fitting the Luttinger parameter from the Friedel analysis Eq.(13) of the B -orbital density in the second incommensurate regime, for $N = 240$ sites ($L = 80$ unit cells) and different fitting windows. (a) Effective exponent K_{eff} and (b) coefficient of determination R^2 versus U . Fits retain $s + 1 \leq i \leq L - s$, with $s = 8, 10, 12, 15$ corresponding to the windows 9–72, 11–70, 13–68, and 16–65, respectively. Identical windows are used at every U .

Relevant Interaction



$$H_{LL} = \frac{v}{2\pi} \int dx \left[K(\partial_x \theta)^2 + \frac{1}{K}(\partial_x \phi)^2 \right],$$

$$H_{\Delta} \propto g_{\Delta} \int dx \cos(2\theta),$$

$$H_u \propto g_u \int dx \cos(4\phi - qx).$$

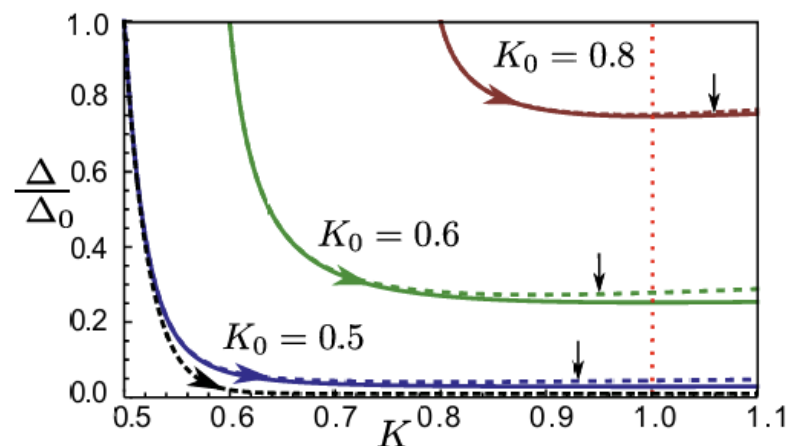
$$\frac{d \ln K}{dl} = \frac{\delta^2}{2K} - 2Ky^2,$$

$$\frac{d\delta}{dl} = (2 - \frac{1}{K})\delta, \quad \frac{dy}{dl} = (2 - 4K)y,$$

For $K < 1/2$ the umklapp term is relevant and superconductivity irrelevant, leading to a Mott phase, whereas for $K > 1/2$ the opposite is true and the system is superconducting. Away from half filling, the Umklapp term becomes strongly oscillated, we can set $y=0$, to solve the KT equation:

$$\delta(l) = 4a\Delta(l)/v_F \text{ and } y(l) = U(l)a'/\pi v_F.$$

$$\Delta(l) = \Delta_0 \frac{\sqrt{8[K(l) - K_0] - 4 \ln[K(l)/K_0] + \delta_0^2}}{\delta_0 \exp[l]},$$



这里的几个参数 Δ 代表了p波超导配对项，也代表了超导能隙。 K 代表了前向散射， $K > 1$ 是吸引相互作用， $K < 1$ 代表了排斥。总体上来看，RG流上对于 $K_0 > 1/2$ 时， K 都会流向强吸引区域， Δ 会减小到一个有限值，有限的超导能隙保护了majorana,这是一个拓扑超导相。对于 $K_0 < 1/2$ 时，RG会流经 $K = 1/2$,而且流经 $K = 1/2$ 时对应GAP最小的时候，从公式91代入 $K = 1/2$, $\Delta = 0$ 看出，如果 $\delta_0 > 2\sqrt{2K_0 - \ln(2K_0)}$ ，在此时GAP也不会闭合，也就是在 $K_0 < 1/2, \delta_0 > 2\sqrt{2K_0 - \ln(2K_0e)}$ 系统仍然保护majorana。在理想情况下，这两种RG流都会流经 $K = 1$ 这个Kiteav Chain的点，呈现明确的majorana。下一个区域对应 $K_0 < 1/2, \delta_0 < 2\sqrt{2K_0 - \ln(2K_0)}$ ，这个区域中最终都会流向 $\Delta = 0$ ， $K_0 < K < 1/2$.这是一个Luttinger液体相,系统gapless,关联函数power decay。