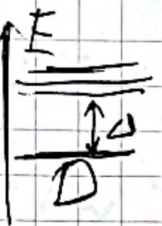
Introduction to Topological order.

1. Gapped Quantum Many-body system.

对基态 

$D = 1$, SPT

D = finite { SSB / Topological order.

D: GS $|\psi_i\rangle$ $i = 1 \sim D$

存在 local operator. 可以区分不同 GS $\langle GS_i | \hat{O}_a | GS_j \rangle$

可表明 D = finite $= \delta_{ij} C$

If $C = C_i$ 为 SSB; 若为 Topological order $C = 1$.

$\langle GS_i | \hat{O} | GS_j \rangle = \delta_{ij}$ 无法区分 (i, j) GS 哪个.

需用 Non-local operator. $(\hat{O}_{ij})$.

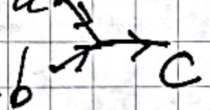
$D = \infty$, = instable; New many-body systems.

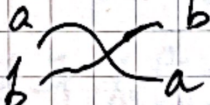
= Goldstone "破缺对"

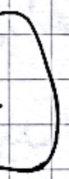
2. Anyon: correlation time finite $\overset{?}{\Rightarrow}$ length finite(3).

, with local excitation. (2-D ? particle-like 

这些 local excitation 并不能用 local operator 产生

with (fusion) properties 

with ? (braiding properties. 

? 

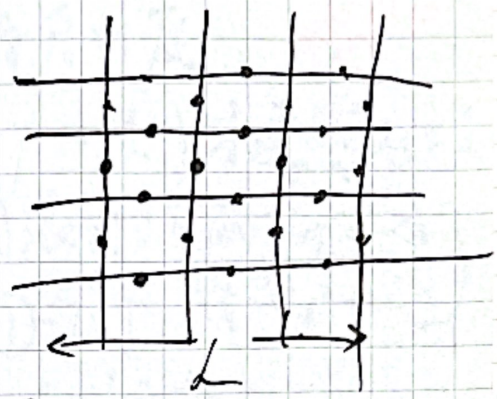
? Example: Spin-Liquid: FQHE.

? Model: Kitaev toric-code; Levin-Wen.

3. Toric code: (Kitaev. 1997). Z_2 gauge. PBC

Link has qubit 或 $|\uparrow\rangle$; $|\downarrow\rangle$.

Hilbert space: 2^{2L^2}

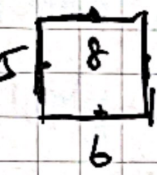


$$H = \sum_v A_v - \sum_p B_p.$$



$A_v = X_1 X_2 X_3 X_4$ 顺序不重要. vertex

X: (pauli X).



$B_p = Z_5 Z_6 Z_7 Z_8$. plaquette.

Z: (pauli Z).

check satisfies: $[A_v, A_{v'}] = 0$ $(\forall v v' p p')$.

$[A_v, B_p] = 0$ $[A_v; H] = 0$.

$[B_p, B_{p'}] = 0$ $[B_p; H] = 0$.

with eigenvalue a_v, b_p.: 所有本征值为 $\{a_v\}; \{b_p\}; \mu$

$$H|\{a_v\}, \{b_p\}, \mu\rangle = -\left(\sum_v a_v + \sum_p b_p\right)|\{a_v\}\{b_p\}\mu\rangle$$ (unknown)

A_v operator has $a_v = \pm 1$ number of $\{a_v, b_p\}$ is

$A_v = L^2$. $B_p = L^2 \Rightarrow 2L^2$ 填满整个 Hilbert space.

对称性 A_v, B_p 足够大以至于不需要集合(μ)了 wrong

[GS: $(a_v = b_p = -1)$ Gap = 2.]



(A_1, A_2) 不独立.

However: A_v not independent of v

$\prod_v A_v = 1$ 为 constraints.

$\prod_v A_v = (X_1)^2 (X_2)^2 \cdots (X_{L^2})^2 = 1 \times \cdots \times 1 = 1$

So we have: $\prod_v a_v = 1$; $\prod_p b_p = 1$

u 就带来了此二限制的缺失的自由度.

u 4重简并. ($a_v = -1$ 必有 $a_{v'} = -1$) 才使 $\prod_v a_v = 1$

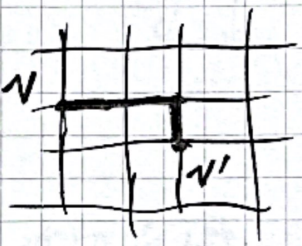
从 $a_v = 1$ (Gs) 到 $(a_v = -1)$ 的激发. 或 $(b_p = -1$ $b_{p'} = -1)$

Actually [Gap = 4] $\begin{matrix} 1\to -1 \\ 1\to -1 \end{matrix}$ > 4 (D=4).

为 $\{|Gs\rangle ; e|Gs\rangle ; m|Gs\rangle \ em|Gs\rangle\}$.

用什么算子才能代表 $\{a_v = -1 \to a_v = 1 ; a_{v'} = -1 \to a_{v'} = 1\}$ 的激发. local operator not work!

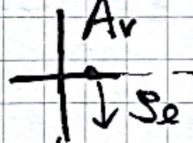
Apply String: $\prod_z^z \sigma^z$ S_e :



$S_e|\Psi_{GS}\rangle = |\Psi_{ex}\rangle$

check: $A_v|\Psi_{ex}\rangle = A_v S_e|\Psi_{GS}\rangle = -S_e A_v|\Psi_{GS}\rangle$

$= -|\Psi_{ex}\rangle$

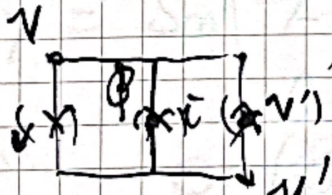


chose - 个 site. 则 $A_v \cdot S_e$ 反对易.

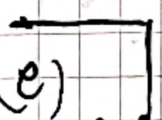
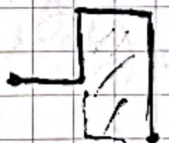
e excitation: 

boson $(Z_v Z_{i'} \cdots Z_{v'})$ follow string.

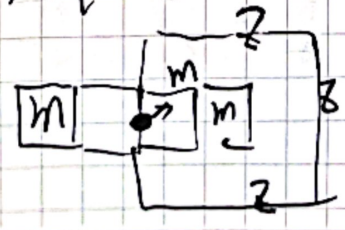
m excitation: 在 Link 上面.



boson $(X_v X_i \cdots X_{i'} X_{v'})$ follow string.

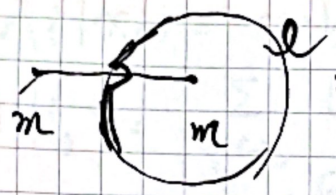
String:  Also 为  则 exe = 1

$exm = \varepsilon$ → fermion $e: (\prod_N A_N)^2$ $m \times m = 1$

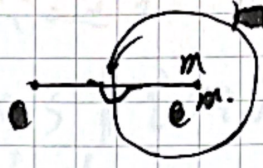


有且仅有1个交点.

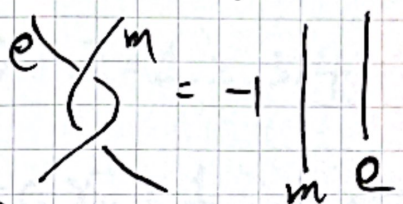
$S_e S_m|\Psi_{GS}\rangle = -S_m S_e|\Psi_{GS}\rangle$

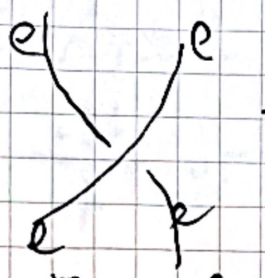


$= e^{i\pi}$



$\theta_{em} = \pi$.



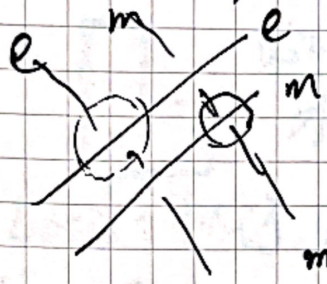


$= e^{i\theta_e}$



$\theta_e = 0$

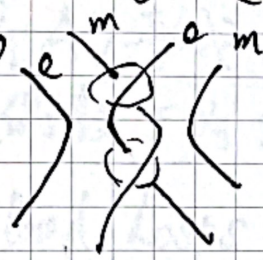
self-statistic.



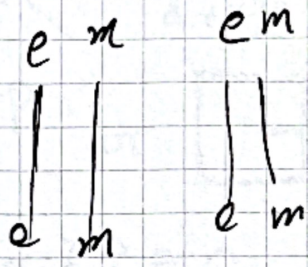
$\theta_e = 0$

$= \theta_m$

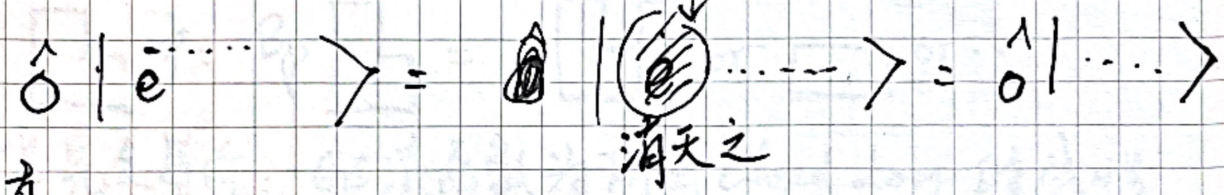
move两交叉点...



$= e^{i\theta_e}$



Assume exists $\hat{O}$ local.



则有:

$$\hat{O} S_m |\psi_e\rangle = \hat{O} \cdot e^{i\theta_{e,m}} |\psi_e\rangle = e^{i\theta_{e,m}} |\psi_e\rangle.$$

and:

$$S_m \hat{O} |\psi_e\rangle = S_m |\text{vacuum}\rangle = 1 \cdot |\rangle$$

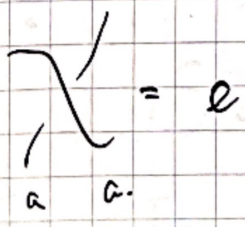
即: $\hat{O}$ 与 S_m:

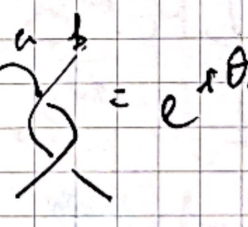
O: local (e)

S_m 无穷大, string多大 (e) 都可以完全

则: $\hat{O} S_m = S_m \hat{O}$ 对易: $e^{i\theta_{em}} = 1$ 意味着 不管(e)周围

$$(e^{i\theta_a}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad (e^{i\theta_{ab}}) = \begin{array}{c|cccc} & 1 & e & m & \varepsilon \\ \hline & 1 & 1 & 1 & 1 \\ & 1 & 1 & -1 & -1 \\ & 1 & -1 & 1 & -1 \\ & 1 & -1 & -1 & 1 \end{array}$$



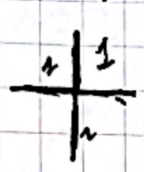


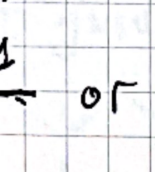
$X_i|x_i\rangle = x_i|x_i\rangle \quad x_i = \pm 1$

$|\psi_\alpha\rangle = \sum_{\{x_i\}} \psi_\alpha(x_1 \cdots x_N)\,|\{x_i\}\rangle \quad \alpha = 1.2.3.4.$
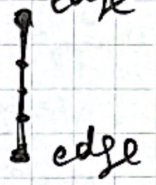
GS-wave function.

GS: ~~Acc~~ 1 $\{a_v = 1\}$, 有: $X_1X_2X_3X_4 = 1$



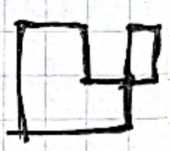
or


or 偶数个为 1



或者为

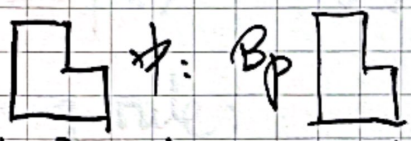

必须所有 A_v 形成 closed Loop 才满足偶数个为 1

or 

basis: $\{x_i\}$ Not closed Loops: $\langle\Psi|\{x_i\}\rangle = 0$

即: 因为 $B_p|\psi_\alpha\rangle = |\psi_\alpha\rangle$ 各种 B_p operator.

B_p 认为是 string hopping operator. 在 A_v 生成

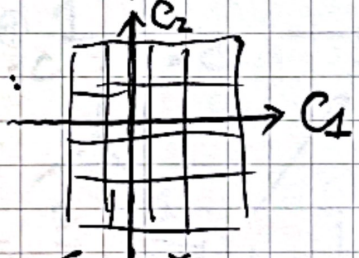


此些完全等价: GS波函数为所有 closed Loop 的线性组合. 且同 $B_p = 1$ 线性组右系数全同.

GS: $|u_1, u_2\rangle = \frac{1}{A}\sum_{\{x_i\}} |x_1 \cdots x_N\rangle$.

with constraints: (1) $A_v = 1$

(2):



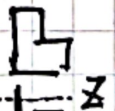
$\prod_{i\in C_1} x_i = u_1 = \pm 1$

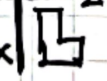
$\prod_{i\in C_2} x_i = u_2 = \pm 1$

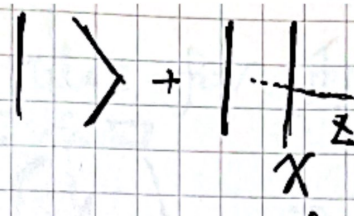
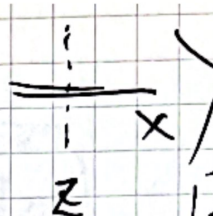
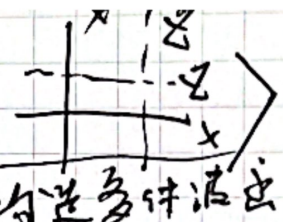
此四重简并

operator: $S^{C_1} = \prod_{i\in C_1} x_i \qquad [S^{C_1}, H] = 0$

$S^{C_2} = \prod_{i\in C_2} x_i \qquad [S^{C_2}, H] = 0$

GS_1:  GS_3:  除非 closed string 首尾到两边

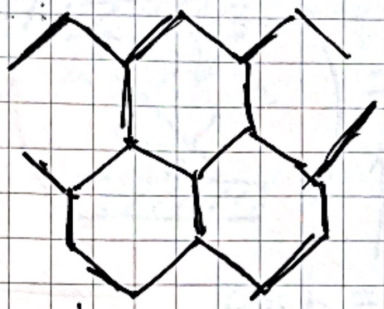
GS_2:  GS_4  的 Line 构成 GS

GS: $|\ \rangle + |$  $\rangle + |$  $\rangle + |$  $\rangle$

找寻构造多体波函数之规则

4. String-net model (Levin-Wen).

Input fusion category $\mathcal{C}$ $\xRightarrow[\text{LW}]{}$ output: topological order: $\mathcal{Z}(\mathcal{C})$



string: N_0 ( )

dual $\bar{a}$ a ( )

Basis: $|\{\text{string configuration}\}\rangle$ $\bar{b}$ b ( )

之前. 仅一种 string. 权重都一样. $\frac{1}{A}$.

现在. $|\Psi\rangle = \sum_x \Psi_x |x\rangle$ configuration

braiding rule:

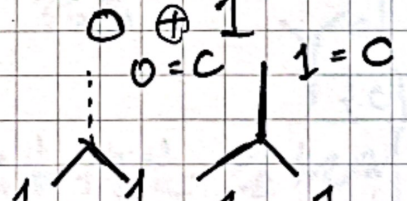
$\mathcal{C}$: $\{a=b, a\neq b\}$ ...

$$a \times b = \sum_c \delta_c^{ab}\, c$$

eg. toric code:

$1 \times 1 = 0$ 两个 Loop 消去 $B_p^2 = \hat{I}$

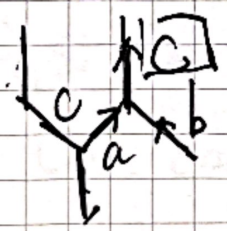
其它:

$1 \otimes 1 =$ 

而 toric code 中的规则即:

$A_v = B_p = 1$

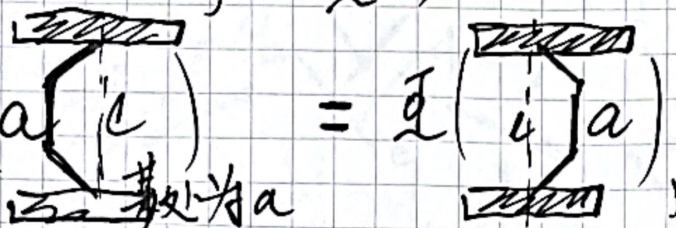
"$A_v \times A_v$" 结果为: | + |



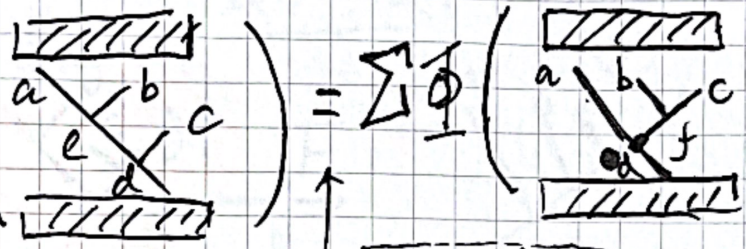
什么 configuration 允许呢?

即下面为 (a,b) 的话. 上面 c 须满足 braiding rule.

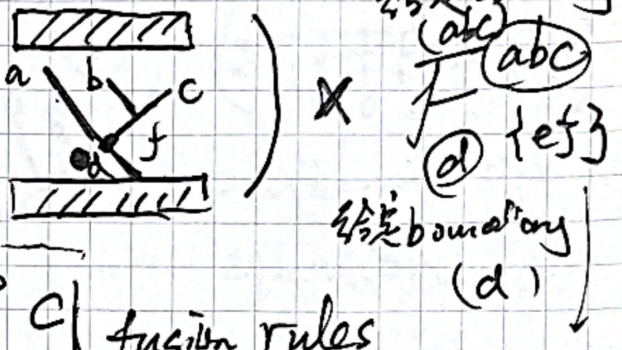
local rules for $\Psi(x)$

(1) 

rule: 直线扭曲两端当a

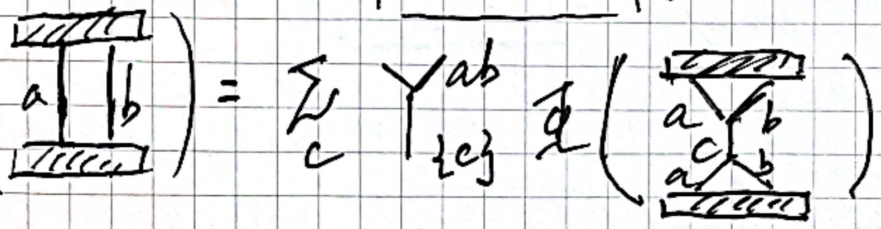
(2) 

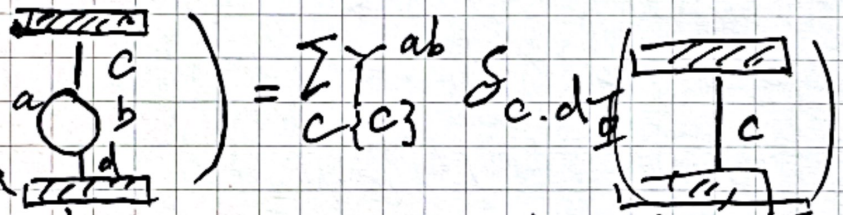
给定boundary



$a \times b = \left| \sum_c \delta_c^{ab} c \right|$ fusion rules.

对ref的线性变换

(3) 

(4) 

从一个没有string的lattice出发. 生成任意一个string configuration的波函数. 对于(找基矢) 这些string configuration 性组合出 $|GS\rangle$. 而系数由这些 braiding rules 给出.

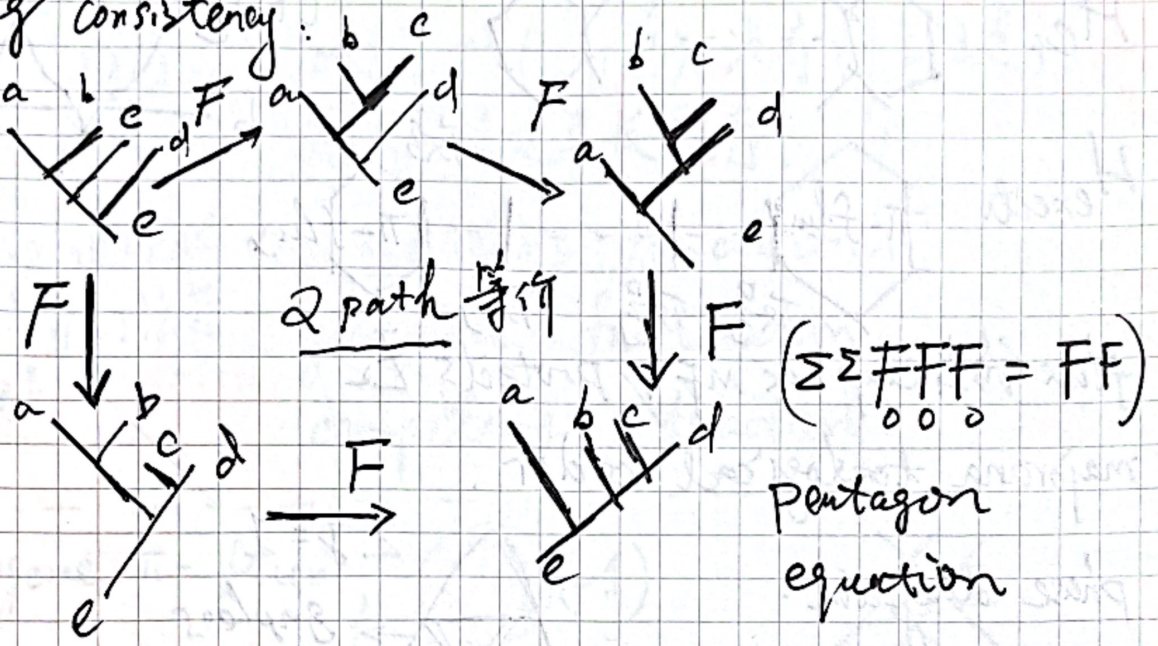
$$Y_{\{c\}}^{ab} = \sqrt{\frac{d_a d_b}{d_c}}$$

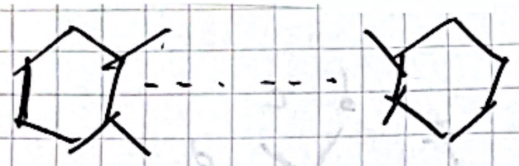
从波函数 G_S 出发构造一套规则, 再从构造规则中找出 Hamiltonian.

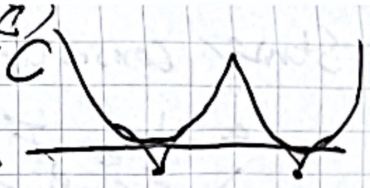
One can prove: exists: $H = \sum_i H_i$

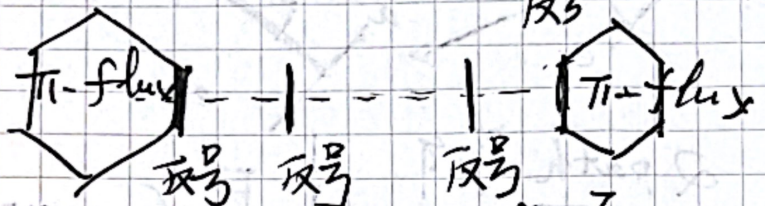
s.t. $H|\Psi\rangle = E|\Psi\rangle$ ↳ We construct

String consistency:



H_c: 

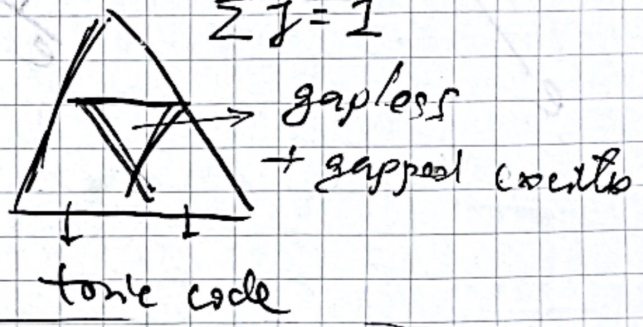


H_{excite} 

flux 给出 Gap. << MF. protects Z_2

majorana topological order.

phase diagram:



Spin dynamics: 不同 flux 构型正交

Jork 格点激发. $\sigma_j^m = i b_j^m c$ $\sigma_k^n = i b_k^n c$

$\langle GS | b_j^m b_k^n | GS \rangle = \delta_{mn} \delta_{k,j+\hat{m}}$

反最近邻关联的 Spin

c-fermion van-hove 奇点提供比热小峰

b-fermion 平带提供比热大峰.

Thermal-Hall. $H^{(3)}$ (力的旋向) 故次近邻 hopping 打开 Gap.

如同 p+ip Superconductor. Chern number $\neq 0$. $\Rightarrow$

Top 非量子计算.

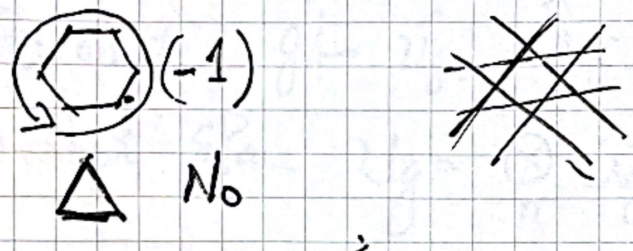
July. 16th

QSL detection: 比热 T^1 (or $T^{2/3}$) 没有峰 (相变)
有纠缠 未建立长程序.

eg. Kagome 磁位 数多 4 比较简单 强.

with strong fluctuation.
geometric frustration.
competing interaction.

Kagome π-flux



做 $\Delta \xrightarrow{\text{gauge}} {}^{-1}_{-1}\Delta_{-1}$ + Translation 不变

choose one (U(1)) gauge. MF 无平移对称性.
但做 gauge transformation. 平移对称性恢复.
MF 下不变等价的 PSG 在固定 IGG 下代表了 Spin Liquid 的种类.

eg. Z_2 SL with translation:

$G_x(i)G_x(i-y) = \pm 1$
$= 1$. 0-flux
$= -1$ π-flux

Kitaev honeycomb. $b^x \cdot b^y \cdot b^z \cdot c \cdot \zeta^{H \cdot W_p}$

$|GS\rangle$ 激发 π-flux 下 $b_j^z |GS\rangle$ = 本征值负 1. 作用把 j 处反号. b 作用

C 作用: 产生 particle-hole 激发

$\delta^m = i b^m \times C$ 从 1 → -1 产生 1 对 π-flux.

即: Spin-flip = 2 π-flux + 1 majorana

即. $|GS\rangle$ 无能隙, 激发有能隙

Gapped system: when $L\to\infty$. $\begin{cases}\text{GS的Gap}\to\text{const.}\\ g_L\to g\ \text{简并度}.\end{cases}$

至则为Gapless: ($g\to\infty$; 连续谱).

when $g=1$; invertible system. ($L\to\infty$ PBC)

no intrinsic topological order.

symmetry G: For $\mathcal{H}=\bigotimes_n \mathcal{H}_n$ ↷ $\hat{U}_{g,n}$

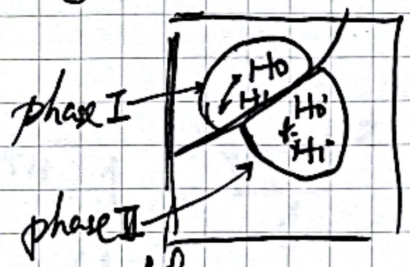
$\hat{U}_{g,n}$为通用原子, G: onsite: $g\mapsto \hat{u}_{gi}$ $\hat{U}_{g_1}\cdot\hat{U}_{g_2}=\hat{U}_{g_1g_2}$

ie. G: go back to Hilbert space. $\hat{U}_g=\bigotimes_n \hat{u}_{g,n}$

G-classification. Assume H_0 & H_1 in the same G-invertible phase $[H_0,G]=0$. $H_0\sim_G H_1$

并有$\exists s_{\in[0,1]}$ $H_{s\in[0,1]}$ is G-sym invertible.

即H_0. H_1同相. 连通.



We want to find $\hat{I}$.

(i) $\hat{I}(H_0)=\hat{I}(H_1)$ if $H_0\sim_G H_1$.

(ii) $\hat{I}(H_0)\neq\hat{I}(H_1)$ if $H_0\not\sim_G H_1$

对G-invertible = topological order"

按照$\hat{I}$完成G-分类

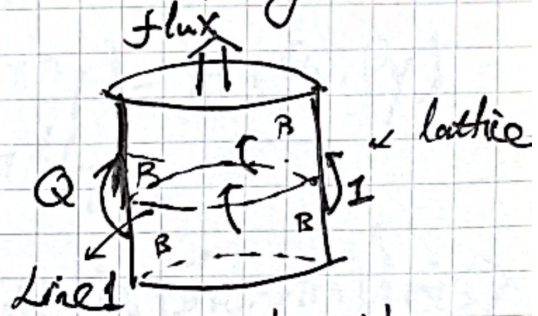
$\hat{I}$离散. (i)自动满足. 因为 $\hat{I}(H_s)=?$

H_s连续; I离散. 离散取值的连续函数为常值函数. 故 $\hat{I}(H_0)=\hat{I}(H_1)$.

Laughlin's Argument. $d=2$, $G=U(1)$ $g=1$.

we have flux $\Phi(t)=2\pi\frac{t}{T}$

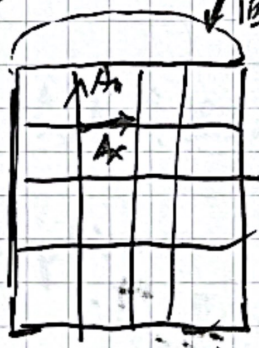
从 $0\to t$ 跨过 Line 1 的从下到上的电量. $\hat{I}=Q(0\to T)/-e$



Put in a lattice:

在连续中. 有 $\vec{A}$ 矢势: $\longmapsto A$

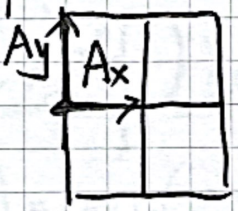
lattice model 中: A 生活在 Link 上



Pierels substitution: $c_i^\dagger c_{i+1}$

$e^{\hat{i}A(i,i+1)}$.

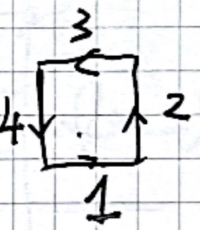
先验: I 离散



$$\begin{cases}\sum_{r\in L_x} A_x(\vec{r},t)=\Phi(t) \\ \sum_{r\in L_y} A_y(\vec{r},t)=0\cdot\pm2\pi\cdot\pm4\pi\end{cases}$$

$(N_0.\text{flux}).(e^{iA})$.

satisfies: $e^{iA_1}e^{iA_2}e^{-iA_3}e^{-iA_4}=1$



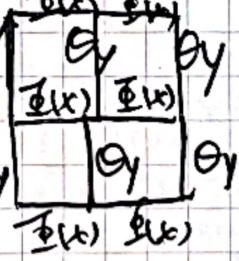
即无旋场等入面内. — flatness.

总可以对 Hamiltonian unitary transformation. st.

$$A_y(r,t)=\theta_y\ (\forall r.)\qquad \theta_x=\Phi(t)$$

$A'(A_{xr_0}, A_{yr_0}\cdots A_{xr}A_{yr})$ 总与 $A(\theta_x.\theta_y.\theta_x.\theta_y-\theta_x\theta_y)$

$=A(\theta_x.\theta_y)$ 等价. 存在么每一个么改写一点 A. 终改为任意的 gauge



$H(A') = U\, H[\theta_x, \theta_y]\, U^\dagger$

PBC 给出: $\hat{H}(\theta_x + 2\pi n_x, \theta_y + 2\pi n_y) = H(\theta_x, \theta_y)$

with $\boxed{\theta_y = 0 \bmod 2\pi}$ $\theta_x = 2\pi t/T$

演化 $|GS\rangle \xrightarrow{0\to T} \mathcal{T}\left[\exp\left\{-i\int dt\, H(\vec{g}(t)\cdot 2\pi\hat{n})\right\}|GS\rangle\right]$

Laughlin's Assumption:

$H(\vec{g}(t)\cdot 2\pi\hat{n})$ 在 Volume 中始终保持为 Gapped.

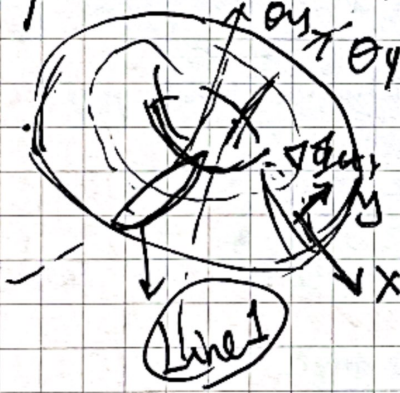
绝热近似 + Laughlin's Assumption.

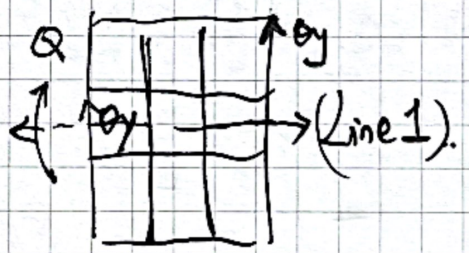
$$|\Psi\rangle = \text{phase} \times |GS\rangle$$

这个 phase 由 Berry phase 给出:

$$|\Psi\rangle = \exp\left[-i\int_0^T E_{g.s}(t)\,dt\right] \times \exp\left(i\gamma^{[n]}_{Berry}\right)|GS\rangle$$

Now. $\theta_y = 2\pi n$., 此上亦为 $\exp(i\gamma^{[0]}) = \exp(i\gamma^{[1]})$





总电荷 AB phase: $(Q\theta_y)$

$\gamma^{[n]} - \gamma^{[0]} = 2\pi n Q$ ($\Psi = \int_\Omega$ 电荷 $\times A$).

而: $[2\pi n Q] \in \mathbb{Z}$ for $\forall n$. $\Rightarrow \underline{Q \in \mathbb{Z}} \to \boxed{I \in \mathbb{Z}}$

($\gamma^{[n]} = \gamma^{[0]}$) 电荷不能半个跳上去.

(垂直于实际出的部分.

即 θ_y 部分提供 (AB) phase 即 Berry phase.)

$$\text{AB phase} = Q\theta_y = \gamma^{[n]}_{Berry} = \gamma^{[0]}_{Berry}.$$

并且. 若 H_0 给出 I_0. H_1 给出 I_1; I_S 对称演绎等价.

更为 $I_0 = I_S = I_1$ [不唯一]

做 $H(A') = UHU^+$ gauge fixing 改变了：

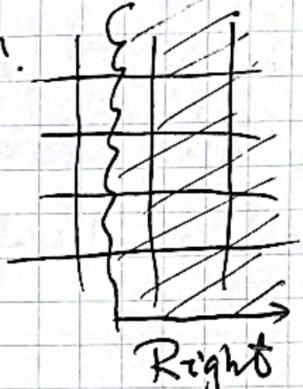
$\nabla \times \vec{A} = 0$ ⋮ ⋯ ⋮ ⋯ ⋮ ⋯ ⋮

$c_i^+ c_j \to c_{g_i}^+ c_{g_j}$

$c_m^+ c_m \to c_n^+ e^{iA_{nm}} c_m$

对于 U：对右侧做 gauge transformation.

$U_R(e^{i\theta_x}) H U_R^+(e^{i\theta_x})$.



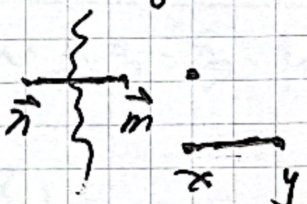
在 OBC 下：$U = \exp(i\theta_x c_x^+ c_1)$.

如右图，c_n 不变，$c_m^+ \to (c_m^+ e^{i\theta_x})$

满足 $c_n^+ c_m \to c_n^+ e^{iA_{nm}} c_m$.

若皆在右侧：$c_y^+ c_x \to c_y^+ c_x$

$c_y^+ \to c_y^+ e^{i\theta_x}$ $c_x \to c_x e^{-i\theta_x}$



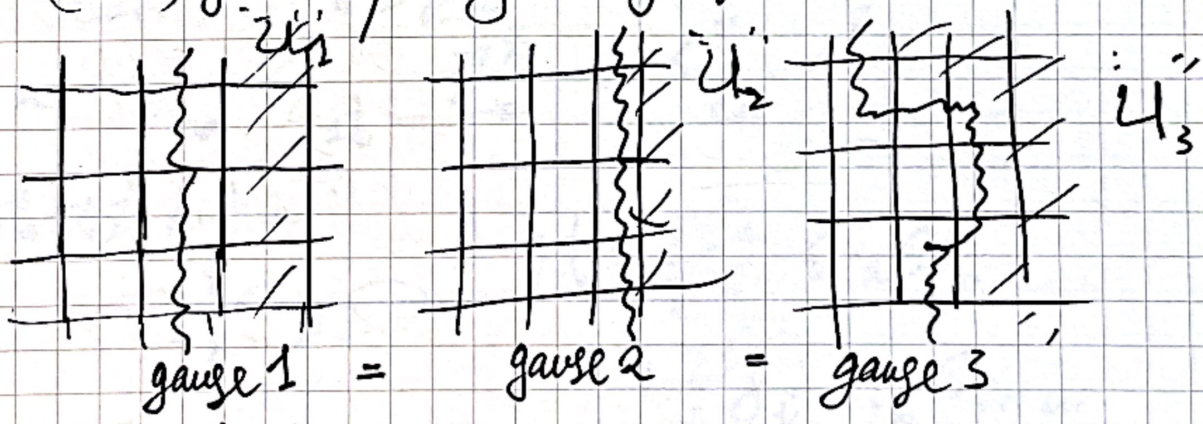
取该线仅该线上有 phase：取一条线 $\Leftrightarrow$ 对右 Hamiltonian 做 Unitary transformation；~~左侧同理~~ θ_y 同理；

即做 Unitary transformation $\Leftrightarrow$ 产生一系列的 θ_x, θ_y.

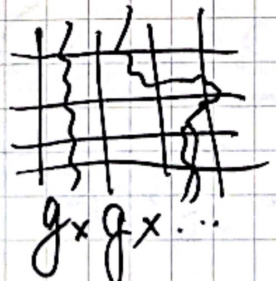
[Flat-U(1) principle]

最后，close the lattice by PBC.

因此：(~~U+U~~) global symmetry $\Leftrightarrow$ gauge invariance.



$=$



Laughlin's Assumption generalized

$H(g_1, g_2, \cdots)$ is also gapped with a unique $|GS\rangle$ if H is invertible. $(H_0 \Leftrightarrow_{G} H_1 \quad (g=1)$

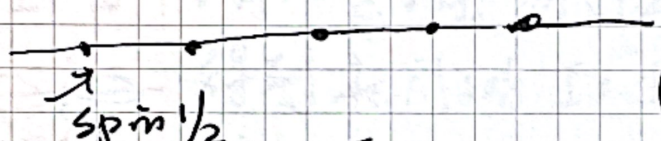
即: $(H_0 \Leftrightarrow_{G_1 \cdot G_2} H_1)$

$H(g)\ g\in G$ 亦可为 $H(g_1, g_2)\ g_1 \cdot g_2 \in G \quad H(g_1 \cdot g_2)$.

对于离散群G. G-background field vs flat

in continuum limit

1-d boson SPT: $Z_2 \times Z_2$ Abelian. $g_1 g_2 = g_2 \cdot g_1$.



$(Z_2^x \cdot Z_2^z)$ Abelian.

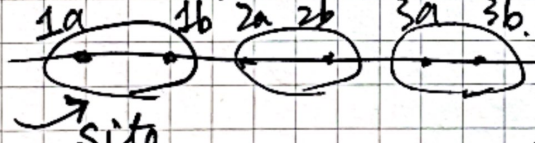
$Z_2^{绕z} \to \exp(i\frac{\sigma^z}{2}\pi) \not\propto \sigma^z$

$Z_2^{绕x} \to \sigma^x$

$\updownarrow$?

$(\sigma^x \cdot \sigma^z)$ 反对易

差3一个负号. 则一个"site"2个spin 1/2"满足, 记.



We should have even number of Qubits.

Now. 挑出绕x转$\pi \in Z_2^x$

$Z_2^x \to \sigma^x_{1a} \cdot \sigma^x_{1b} \qquad Z_2^z \to \sigma^z_{1a} \cdot \sigma^z_{1b}$.

只能切在link上.



$$H = \sum_i J_i S_{ia} S_{ib} + \sum_i J_i S_{ib} S_{i+1\,a}$$

只有 $\widetilde{J}_{i_0} S_{i_0 b} S_{i_0+1\,a}$ 变了

只有这个 Rotate π 绕x

$$\hat{U} = \widehat{J} S^x_{1b} S^x_{1a} + \widetilde{J} S^y_{1b}(-S^y_{1a}) + \widehat{J} S^z_{1b}(-S^z_{1a})$$

定义 $\hat{I}$：$H \rightsquigarrow H(\{\sigma_x\})$ 给 x 转下

$|GS\rangle \rightsquigarrow |\widetilde{GS}_x\rangle$ $U_{\Phi_z} = \prod_{n=1}^{L} \sigma_{na}^z \sigma_{nb}^z$ 转半圈

$$\mathcal{I} \equiv \frac{\langle \widetilde{GS}_x | U_{\oint_z^{\pi}} | \widetilde{GS}_x \rangle}{\langle GS | U_{\oint_z^{\pi}} | GS \rangle} \quad \in \{\pm 1\}$$ 只可能

对于 H，可以计算 $\hat{I}$.

① 存在：$H_1 = \sum_{i=1}^{L} \vec{S}_{na} \cdot \vec{S}_{nb}$ $I = 1$.

(PBC) $H_2 = \sum_{i=1}^{L} \vec{S}_{nb} \cdot \vec{S}_{(n+1)a}$ $I = -1$

对于 $I=1$ $\forall H \sim_G H_1$ 都可由上面 unitary transformation 得到相同的 $I=1$

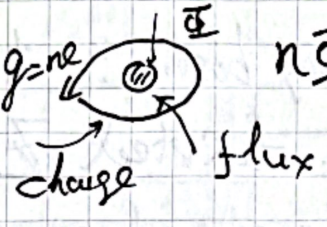
$I \Leftarrow [THT^{-1}]$ H 给出 I；T 平移 1 晶格矢，$TH\frac{1}{T}$ 给出 $-\hat{I}$；在 $\mathbb{Z}_2 \times \mathbb{Z}_2$；spin 1/2；$H = T\frac{1}{H}T^{-1}$ Not invertible

Discrete Gauge theory: $Z_n: \{1 \cdots n-1\} \subset U(1), \{e^{i\frac{2\pi}{N}k}\}$

θ_{AB} (AB-phase) $= n \cdot \frac{2\pi \Phi e}{h}$

flux: $0 \cdot \frac{2\pi}{N} \cdots \frac{2\pi}{N}(n-1)$

charge: $0 \cdot 1 \cdot 2 \cdots N-1$



Def: anyons in Z_n gauge theories: 让个e绕了个flux为anyon.

$e^i \cdot m^j \equiv (i,j)$ $\quad N: (1-j)$

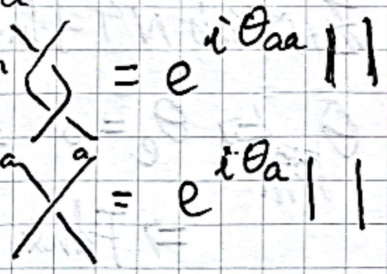
eg. $n=2$ $(0,0)$ $(0,1)$ $(1,0)$ $(1,1)$: 1. e. m. ε

群元乘法 $(e^i \cdot m^j) \times (e^k \cdot m^l) = e^{i+k} \cdot m^{j+l}$

群单位元 $(e^1 \cdot m^1)$: AB effect 在 Z_n 而非 $U(1)$ 中: $\frac{2\pi}{N} \neq \theta_{em}$

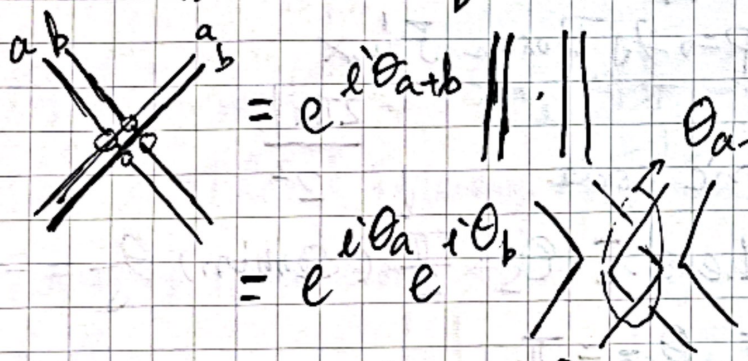
Assume: $\theta_e = 0$ (bosonic matter)

$\theta_m = 0$ $(F=1)$ Flux Boson

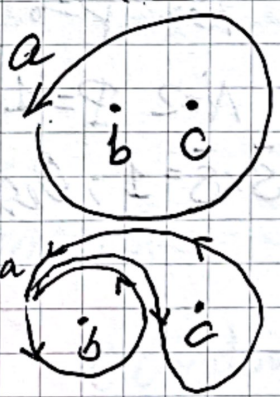


with properties (1) $\theta_{aa} = 2\theta_a$

(2) $\theta_{a+b} = \theta_a + \theta_b + \theta_{a,b}$



$\theta_{a,b+c}$ a绕b+c转1圈



(3) $\theta_{a,b+c} = \theta_{a,b} + \theta_{a,c}$

Now, from (1 e.m.ε): θ_{em}.

to $\theta_{e^i \cdot m^j ; e^k \cdot m^l} = \theta_{e^i ; e^k} + \theta_{e^i \cdot m^l} + \theta_{m^j \cdot e^k}$

"+" $\quad + \theta_{m^j \cdot m^l} = \frac{2\pi}{N}(il + jk)$

$\theta_{e^i m^j} = \theta_{e^i} + \theta_{m^j} + \theta_{e^i \cdot m^j} = \frac{2\pi}{N}(ij)$

Example: $\mathbb{Z}_4$: of anyons: 16. when $\theta_a \neq 0$

How many bosons? $1, e, e^2, e^3, m, m^2, m^3, e^2m^2$

Example: twisted $\mathbb{Z}_4$ gauge theory:

$F = \gamma_3(a,b,c) \in H^3(G, U(1))$ $\nearrow \mathbb{Z}_N$

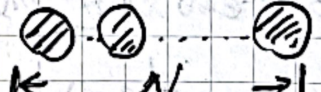
$FF = \overline{FFF}$ $P = 0, 1, \cdots N-1$

$\gamma_3 = \exp\left\{ i\frac{2\pi P}{N} a\left[b + C - (b+c) \bmod N \right] \right\}$

只改变了 $\theta_m \neq 0$. $\theta_e = 0$ $\theta_{em} = \frac{2\pi}{N}$ 皆不变.

For $F = \gamma_3(P)$ 从上面 properties 推论: $\theta_{na} = n^2\theta_a$

$\theta_{km} = k^2\theta_m$ Take: $K = N$.

即存在 N 个 flux 

$\mathbb{Z}_N$ 中的 N 个 flux: $\left(\frac{2\pi}{N} \cdot N\right) = 2\pi$ 不是 flux. 就是 U(1) gauge 就是一个 charge.

$\theta_{\underbrace{a}_{Nm}} \leftrightarrow \theta_e = 0 \Rightarrow K^2\theta_m = 0 \bmod 2\pi$

$\Rightarrow$ Flux $\theta_m = \frac{2\pi P}{N^2}$

即 $F = \gamma_3(P) = \frac{2\pi P}{N^2}$ $P = 0$ 为 Boson Flux

Example: $N=2$ $P=0$ toric code $\frac{2\pi \times}{2^2}$

$N=2$ $P=1$. $1, e, s, \bar{s}$ $\theta_s = \frac{\pi}{2}$ (semion) $\theta_{\bar{s}} = -\frac{\pi}{2}$

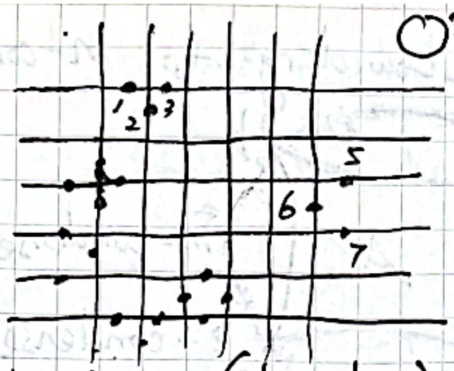
$s \times s = 1$; $\bar{s} \times \bar{s} = 1$; $\theta_{es} = \pi$ $\theta_{e\bar{s}} = \pi$

Gapped boundary:

OBC toric code 在 boundary

选择: $\widehat{A_v} = X_1X_2X_3$. 下面:

$\widehat{B_p} = Z_5Z_6Z_7$ 为 Boundary.



Lattice: $(L \times L)$

Number of qubits: $(L+1)^2 + L^2$

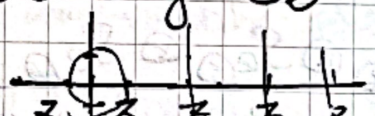
Number of $A_v = L\cdot(L+1)$

Number of $B_p = L\cdot(L+1)$

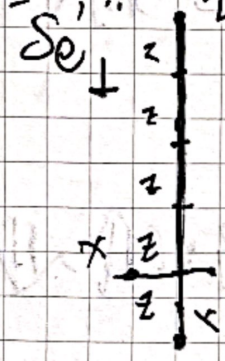
$(L+1)^2 + L^2 - 2L(L+1)$

$= 1$ Degeneracy $= 2^1 = 2$

如同上文 String = Se''



$=Se''$ 与 A_v. B_p 都对易 与补对易. 可用 $=Se_{//}''$ label 基态

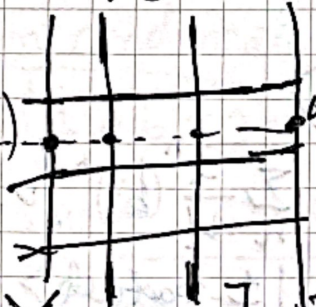


$=Se_{\perp}''$ 不对易

同理 $=Sm''$ (X) 与 $=Sm_{\perp}''$ 对易 $=Sm_{//}''$ 不对易. 而 $Sm \cdot Se$ 不对易. 取一个 label Degeneracy 即可 $\{Sm_{//}=1;\ Sm_{\perp}=-1\} = 2$

Boundary condition 决定了 Degeneracy!

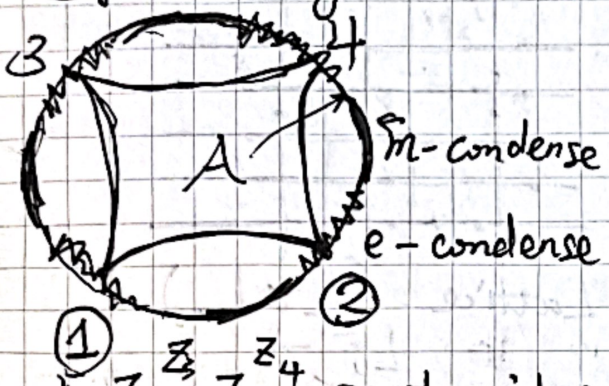
选择:



$\widehat{Sm} = [\widehat{m\,String} \times X_{Boundary}]$ $X_{boundary}$ 让 m string 消失了

在 bulk 中不可能有 local operator 产生 local annihilation of $\widehat{Sm}$ 激发: 在上面即有 2个 m condensed. 2个 e condensed operator.

选取 boundary A: m-condensed 即 S_m 在 boundary 处为 1



有 Z_1, Z_2 在 ① Z_4 operator 让 e-condense

这一条 string 有 2^1 Degeneracy. 指向"1". 与另外的 2' 指向 (2.3.4) 的 string 简并. 共 2^4. 连接 1~2~3~4 的这连接此4线. 合成 "S_e" $= S_{e①} \otimes S_{e②} \otimes S_{e③} \otimes S_{e④} = 2^4$

由于为 1~2~3~4 closed Loop 首尾相接扣除 2' 简并度余下 $D = 2^{N-1}$ (toric code 中 $N=2$)

$S_1^e = \cdots = S_{N-1}^e = \pm 1$ $S_i^e = ZZ\cdots Z$ $S_i^m = XX\cdots X$

$|GS\rangle$: $|a_1 \cdots a_{N-1}\rangle$ a_1: $\{S_1^e = 1;\ S_1^e = -1\}$: (↑, ↓)

S^e 产生大 spin. S^m 为 spin flip operator.

$|a_1 \cdots a_{N-1}\rangle$ 实际为 2-D spin "→ spin" 为 "Line" $|GS\rangle$ 构成 "surface code"

何为 "condensate"?

BEC GS: $b^\dagger|GS\rangle \equiv |GS\rangle$

coherent State $e^{ab^\dagger}|0\rangle = \sum_{n} \frac{a^n}{n!}(b^\dagger)^n|0\rangle$

$= c_0|0\text{-particle}\rangle + a_1|1\text{-particle}\rangle + \cdots$

ie: BEC condensate: Not conserve Q_N

Now, e-condensate edge state: [|0-particle⟩ + |1-particle⟩]

$\{e: |0\rangle + |e\rangle$ $(S_i^e)[|0\rangle + |e\rangle] = |e\rangle + |0\rangle$ is a Not conserve QN condensate state.

另一方面 —m— e —m— domain wall σ $\sigma \times \sigma = 1 + e$

每个 domain wall ⇔ majorana Zero mode. $|0\rangle \xrightarrow{\gamma^\dagger} |0\rangle$ Zero Energy Excitation

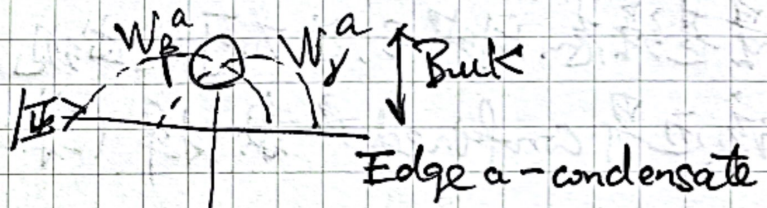
2个 domain wall 相当于一个 charge + 0个 charge.

细节: e-edge 定义在 String 上的 operator 要做 Jordan-Wigner transformation 得到 Fermion operator 的 domain wall MZM

∴ 2种不同边界交点可能有 MZM

Anyon condensation: what kind of anyons can be condensed?

$D = Z(\mathcal{C})$ 必须 $\theta_a = 0$.



$\begin{cases} W_\beta^a |亚\rangle = |亚\rangle \\ W_\gamma^a |亚\rangle = |亚\rangle \end{cases}$ 亦必有:

$\begin{cases} W_\beta^a W_\gamma^a |亚\rangle = |亚\rangle \\ W_\gamma^a W_\beta^a |亚\rangle = |亚\rangle \end{cases}$ $\Rightarrow W_\beta W_\gamma = W_\gamma W_\beta$

而 $W_\beta W_\gamma = e^{i\theta_{aa}} W_\gamma W_\beta \Rightarrow \theta_a = 0$

criteria: Boson condensate is must for anyon-condensate

In general. $\{1, a, b - \} = A$ can condensate.

should. c1) $\theta_a = 0 \quad \forall a \in A$ c2) $\theta_{ab} = 0 \quad \forall a, b \in A$

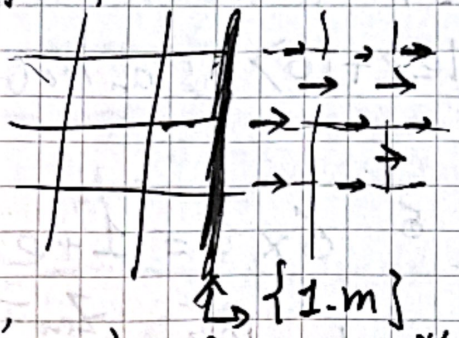
c3) $\bar{A}$ must have non-trival braiding in A

eg. $A: \{1, m\}$ $\bar{A} = (e, a)$ m-condensate

↳ $\{1, m\}(\bar{A})$ Toric code

= confinement

选取辅助场"m-condensed phase"



辅助场为"spin全+x"的态

$\mathcal{H}=\mathcal{H}_L\otimes\mathcal{H}_R$ $\mathcal{H}_R$不对$\mathcal{H}_L$有任何改变 $\mathcal{H}_R^{\otimes_R}\otimes\sum_i\otimes X_i$

$\mathcal{H}=\bigotimes\mathcal{H}=\mathcal{H}_L$

"S_m"没有改变(全正1)可以让m移入右边、$\mathcal{H}$仍为$\mathcal{H}_R$

可以认为右边为"m-condensed phase"

toric code	m-conde
1→1→1	1…→1
m→m→m	1→1
e→e→e	E不变

对于"e"移入右边 $\cdots e\cdots\sum_x(\to)=\{1\text{ or }-1\}$都可以

由于五边开的"→"涨落无极强，故e右移中产生能量基强态，同理e，这不可能出现，故(e,e)必要在该边界confined要拿掉(禁闭)。

PBC: $C^\dagger(L_x+x, y) \equiv C^\dagger(x, y)$

$C^\dagger(x, L_y+y) \equiv C^\dagger(x, y)$

Apply A gauge: $C^\dagger_{x,y} C_{y} \rightsquigarrow [C^\dagger_{(x,1)} e^{i\theta_y} C_{(x,L_y)}]$

Similarly TBC:

$$\boxed{\begin{aligned} C^\dagger_{(L_x+x,y)} &\equiv C^\dagger_{(x,y)} e^{i\theta_x} \\ C^\dagger_{(x,L_y+y)} &\equiv C^\dagger_{(x,y)} e^{i\theta_y} \end{aligned}}$$

即原始 Hamiltonian 无边界条件 $\neq$ flat background gauge field

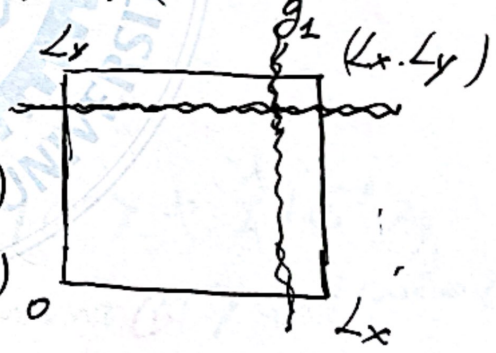
$\Leftrightarrow$ Hamiltonian Applying Tc twisted) BC

$H \rightsquigarrow H(e^{i\theta_x}, e^{i\theta_y}) \Leftrightarrow H(g_x, g_y)$ (x,y)方向做 gauge field.

必有 $g_x g_y = g_y \cdot g_x$ 从 TBC 观点看:

g_x: $\psi(x,y) = g_1 \cdot \psi(L_x+x, y)$ $(g_1 \equiv \hat{I}$: PBC) $(g_1 \equiv e^{i\theta_x})$ TBC.

g_y: $\psi(x,y) = g_2 \cdot \psi(x, L_y+y)$



则: $\psi(0,0) = g_1 \cdot \psi(L_x, 0) = g_1 g_2 \psi(L_x, L_y)$

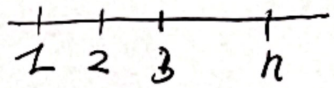
$= g_1 g_2 g_1^{-1} \psi(0, L_y) = g_1 g_2 g_1^{-1} g_2^{-1} \psi(0,0)$

即 $g_1 \cdot g_2 = g_2 \cdot g_1$ 单位群元 $g_1 g_2 g_1^{-1} g_2^{-1}$

Δ (H with A_μ, $F_{\mu\nu}=0$) $\Leftrightarrow$ (TBC with $g_1 \cdot g_2 = g_2 \cdot g_1$)

$H_{non\text{-}trival} = [1 + (-1)^n]\,\delta_n \cdot \delta_{n+1}$

$H_{trival} = [1 - (-1)^n]\,\delta_n \cdot \delta_{n+1}$.



$H_g = [1 + (-1)^n \cdot g]\,\vec{S}_n \cdot \vec{S}_{n+1}$



$H(\delta=1) = H(\delta \to 0^+)$ Gapped $H \Leftrightarrow H_{trival}$ $H \Leftrightarrow H_{non\text{-}trival}$ Gapped

$H(\delta=-1) = H(\delta \to 0^-)$ using perturbed field theory.

对 $H(\delta=0)$ 做微扰

$H_{g<<1} = WZW_1 + h^{\delta}_{int}$ (Heisenberg)

Lattice 对应的均匀 $\delta=0$ 模型是什么？

对每个 site：spin½ $\mathrm{span}\{|\uparrow\rangle, |\downarrow\rangle\} = \mathbb{C}^2$

(对比：spinful fermion：$\mathbb{C}^4$：$\mathrm{span}\{|vac\rangle, |spin\uparrow fermion\rangle, |spin\downarrow fermion\rangle, |\uparrow fermion, \downarrow fermion\rangle\}$)

如何用 $\mathbb{C}^4$ + 约束描述 $\mathbb{C}^2$？

constraint: $c^\dagger_\uparrow c_\uparrow + c^\dagger_\downarrow c_\downarrow = 1$

constraint spinful fermion $\Leftrightarrow$ spin ½.

Is U(1) gauge symmetry apply on $\mathbb{C}^4$ $e^{i\theta\hat{n}}|\mathbb{C}^4\rangle \neq \hat{I}|\mathbb{C}^4\rangle$

$e^{i\theta n}|\text{constraint } \mathbb{C}^4\rangle = e^{i\theta}|\text{constraint } \mathbb{C}^4\rangle \Leftrightarrow |\text{Constraint } \mathbb{C}^4\rangle$

拥有 gauge symmetry，即：Apply symmetry 之后只改变一(相同)相位

U(1) gauge symmetry

原来：$H(\delta)$ 拥有 SO(3) Rotate symmetry. $\vec{S} \mapsto \vec{R}\cdot\vec{S}$

S_n in $|spin\,1/2\rangle_{basis} \Leftrightarrow (c_\uparrow\ c_\downarrow)^+ \frac{\vec{\sigma}}{2}\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$ in $|constraint\ spinful\ fermion\rangle_{bas}$

而 $SO(3) \Leftrightarrow U(2)$

$S \mapsto RS$

$\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} \mapsto U_R \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$ U(2) 中亦包含 U(1) gauge symmetry 在 constraint fermion spinful 中

数学 $U(2)/U(1) = SO(3)$

Fixed theory Apply with Lorentz invariance group is CFT:

$(\omega_0 \omega_1)$ ⇓ Dirac Fermion

$\mathcal{L}_{Dirac(2)} = \sum_\sigma \bar{\psi}_\sigma^{(t,x)} (i\gamma^\mu \partial_\mu) \psi_\sigma^{(t,x)}$ $\quad 4\times 4$ $\quad \psi_\sigma = \begin{pmatrix} \psi_{L\sigma} \\ \psi_{R\sigma} \end{pmatrix}$

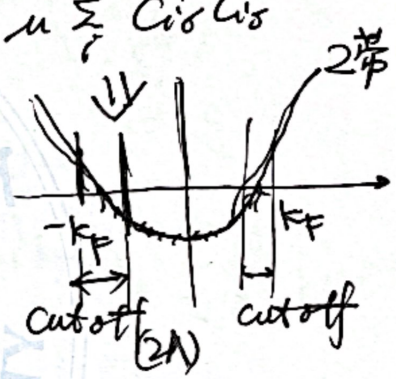
$|\uparrow\rangle, |\downarrow\rangle$ flavor $\quad (\gamma^t, \gamma^x) = (i\sigma^z, \sigma^x) \quad \bar{\psi}_\sigma = \psi_\sigma^\dagger \gamma^t \quad \not{\partial} = \gamma^\mu \partial_\mu \quad \mu = (t, \vec{x})$

= Free Dirac Fermion Field ⇔ Tight Binding $H_0 = \sum_i c_{i+1\sigma}^\dagger c_{i\sigma} + h.c.$ $+ \mu \sum_i c_{i\sigma}^\dagger c_{i\sigma}$

ψ_L, ψ_R 是什么？ $k_F = N_{tot} \cdot \frac{2\pi/L}{4}$

总粒子数 $N_{tot} = L/a$ $\quad \frac{k_F}{(2\pi/L)} \cdot 2 = N$ ↓ $(|\uparrow\rangle, |\downarrow\rangle)$ 2带

$\Rightarrow k_F = \pi/2a$



$\psi_\sigma : (\psi_{L\uparrow}, \psi_{L\downarrow}, \psi_{R\uparrow}, \psi_{R\downarrow})$ 低能等效中：

$c_{j\sigma} = \int_{BZ\,in\,(2\Lambda)} dk \cdot c_\sigma(k) e^{ikx} \quad (x = ja)$

$= \int_{BZ\,in\,Left\,2\Lambda} \frac{d(\Delta k)}{2\pi} c_\sigma(-k_F + \Delta k) e^{i\Delta k x} e^{-ik_F x} + Right$

波动率 $= e^{-ik_F x}\left(e^{ik_F(x+a)} \mp e^{ik_F x}\right)$

$e^{ik_F x} = e^{i\frac{\pi}{2}ja} = e^{(\frac{1}{2}\pi i)\cdot j}$ (c_j) 的关于 j 的跳变源于 $e^{ik_F x}$.

余下的部分就是关于 x 连续的场 $\psi_L(x)$

$c_{j\sigma} = \int_{2\Lambda} \psi_L(x) e^{ik_F x} + \psi_R(x) e^{ik_F x}$

$\psi_L(x) = \int_{2\Lambda} d\Delta k\, c_\sigma(-k_F + \Delta k) e^{i\Delta k x}$

Constrained Dirac Fermion: $\longleftrightarrow$ Spin 1/2 chain.

$\psi^\dagger_{4\times1}\frac{\delta}{2}\psi \longleftrightarrow S$

$\psi_L \mapsto \delta^z\psi_L \quad \psi_R \mapsto \delta^z\psi_R \longleftrightarrow \begin{pmatrix} S^x_i \\ S^y_i \\ S^z_i \end{pmatrix} \xrightarrow{R^z_\pi} \begin{pmatrix} -S^x_i \\ -S^y_i \\ +S^z_i \end{pmatrix}$

$\psi_\delta \mapsto \delta^z\psi_\delta \longleftrightarrow \begin{pmatrix} S^x_i \\ S^y_i \\ S^z_i \end{pmatrix} \xrightarrow{R^x_\pi} \begin{pmatrix} +S^x_i \\ -S^y_i \\ -S^z_i \end{pmatrix}$

$\psi(x) \xmapsto{\hat{T}} ? \longleftrightarrow S_j \xmapsto{\hat{T}} S_{j+1}$ 剧烈变化

$TC_jT^{-1} = S_{j+1}$ 当 $C_j = \psi_L e^{-ikx} + \text{Right}$

$\Rightarrow T(\psi^\dagger_L e^{-ikja} + \psi^\dagger_R e^{ikja})T^{-1} \Rightarrow T\psi_L(ja)T^{-1}e^{-ikja} + T\psi_R(ja)e^{ik}$

$= \psi_L(ja+a)e^{-ik(ja+a)} + \psi_R(ja+a)e^{ik(ja+a)}$.

$\Rightarrow T\psi_LT^{-1} = (-1)\psi_L \quad T\psi_RT^{-1} = i\psi_L$ axial transformation

$\left[\psi(x) \mapsto -i\tau^z\psi(x)\right]\tau^z = \begin{pmatrix} LL & LR \\ RL & RR \end{pmatrix}_z$

因此. $H(\delta=0) \Leftrightarrow \mathcal{L}_{\text{Dirac}\sim2}[\psi,\bar{\psi},a]$ constrained.

而 h_δ 观察 $H(\delta=1)$ 与 $H(\delta=-1)$ 区别:

有 $TH_\delta T^{-1} = H_{-\delta} \quad H = H(\delta=0) + h_\delta$

而 $TH_0T^{-1} = H_0$ 可验. 即 $Th_\delta T^{-1} = h_{-\delta}$

什么 h_δ 满足 $Th_\delta T^{-1} = h_{-\delta}$? 宏的质量项.

$h_\delta \Leftrightarrow \delta\cdot(\bar{\psi}\psi)$

$Z[\delta] = \int d\bar{\psi} d\psi \Big|_{\Omega} da[U(1)] \exp[-S_{Dirac}]$ → 上文得到

需要 Ω gauge field (background flat) $G = U(1)$

TBC ⇒ : $\psi(t,x) = -\delta^x \psi(t, L+x)$

$\psi(t,x) = \delta^z \psi(t+\beta, x)$ ← 虚时 From Hamilton gauge

(之后还要加 $a_{U(1)}$ gauge 变)

$Z = Tr[R_\Pi^z e^{-\beta \hat{H}(R_\Pi^z)}]$

$= \langle \psi_0 | R_\Pi^z | \psi_{N+1} \rangle \langle \psi_{N+1} | e^{-\frac{\beta}{N} H(R_\Pi^z)} | \psi_N \rangle$ → Langrange $\psi(\beta - \beta/N)$

$\times \langle \psi_N | e^{-\beta/N \hat{H}(R_\Pi^z)} | \psi_{N-1} \rangle \langle \psi_{N-1} | \cdots \cdots \times \langle \psi_1 | e^{-\beta/N \hat{H}(R_\Pi^z)} | \psi$

$\langle \psi_0 | R_\Pi^z | \psi_{N+1} \rangle = \langle \psi(0) | \delta^z \psi(\beta) \rangle$

$= \delta[\psi(t=0,x)][\delta^z \psi(t=\beta, x)]$

得到上面的路径积分公式. 边界条件 ~~$\psi(x)$~~ 到据 TBC.

GS of $H(R_\Pi^x)$ 回忆 $\hat{I}_\delta = \langle \widehat{GS} | R_\Pi^z | \widehat{GS} \rangle$

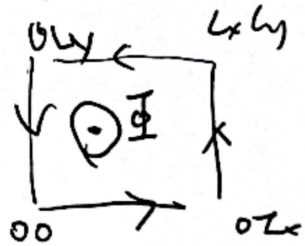
计算方法虚温正负都当得于 GS.

$\hat{I}_\delta = \lim_{\beta \to \infty} Tr[R_\Pi^z e^{-\beta \hat{H}(R_\Pi^z)}]$ 路径积分.

因此. 计算: $\frac{Z_{Dirac}[\delta]}{Z_{Dirac}[I-\delta]} = \frac{I_\delta}{I_{-\delta}}$

↑场论 ↑晶格.

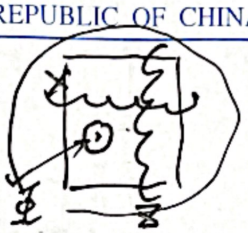
在 TBC 边界下面

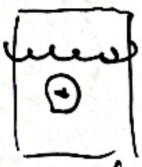


TBC 还需满足 $g_1 \cdot g_2 = g_2 \cdot g_1$. 而 $\delta^x \delta^z$ 不对易?

$Z = \int d\psi d\psi [d[a_{U(1)}]] \sim e^{i\pi} (\delta^x e^{i\pi} \delta^z)$ $\hat{x}: (\psi, a)$.

上下 TBC 再加入规范场 $\psi(t,x) = e^{i\pi} \delta^x \psi(t, -L+x)$. 等价

$\theta = \oint da(u) = \pi \bmod 2\pi$ for:  转一圈 $\delta^x \delta^z$

类似:  ; 

$\theta = 0 \bmod 2\pi$ 转一圈 δ^x $\theta = 0 \bmod 2\pi$

总之: $\theta = \pi I_{nt}\begin{pmatrix} g_1 & g_2 \\ x & z \end{pmatrix} \bmod 2\pi$ 两类 g_1, g_2 的反叉数.

总之可计算 $\frac{Z[\delta]}{Z[-\delta]}$ 在 Hamiltonian TBC + Apply θ for (g_1, g_2) gauge 下面:

$$= \frac{\int d\psi d\bar{\psi}\, da(u) \exp[-(\mathcal{L}_{Dirac} + \delta\bar{\psi}\psi)]}{\int d\psi d\bar{\psi}\, da(u) \exp[-(\mathcal{L}_{Dirac} - \delta\bar{\psi}\psi)]}$$

相变量代换: $\psi = U_z \psi'$ 不改变 Dirac 部分 而此变换即 $T h_\delta T^{-1} = h_{-\delta}$ 由 $(U_z)^2 = \mathbb{1}$

$$= \frac{Z[\delta]}{\int D[\bar{\psi}'\psi'] da(u) \exp[-\int \mathcal{L}_D[\bar{\psi}'\psi'] + \delta\bar{\psi}'\psi']}$$ 两者只差

测度 $\int D[\bar{\psi}', \psi'] \exp\left[i 2\frac{\Pi}{2} \oint \frac{da(u)}{2\pi} \times \underset{(x,y)}{2}\right] = \int D[\bar{\psi}, \psi]$

$$= \exp[-i \oint da(u)]$$

$$= \exp\left[i\pi I_{nt}\begin{pmatrix} g_1 & g_2 \\ x & z \end{pmatrix}\right]$$ 为 Topological order.

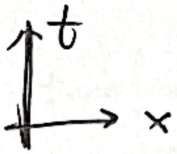
因此: $\frac{Z[\delta]}{Z[-\delta]} = \pm 1$ 如上文所说.

$$= \exp\left[i\pi I_{nt}\begin{pmatrix} g_1 & g_2 \\ y & z \end{pmatrix}\right]$$

- 直接计算构造 Hamiltonian 时 $(g_1 g_2)$ 反叉数可以得到 Topological Invariants.

g_1, g_2 定义在球面上. 无法判断 Z

Z() = Z() 两点, trival



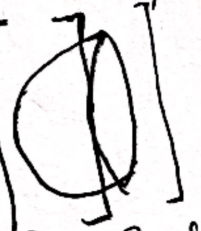
定义在环面上 toric

Z() = -1 = 

定义在有边界球面上

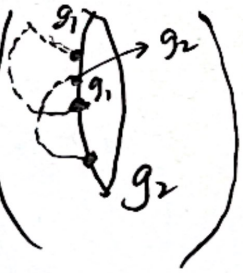
Z () = Z () trival

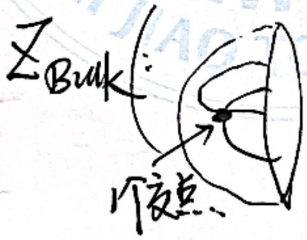
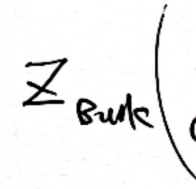
但有边界可区分 Bulk. Boundary.

Z() = $Z_{Bulk} \cdot Z_{Boundary}$

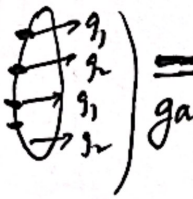
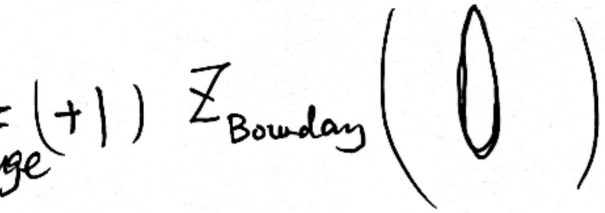
Bulk Boundary

闭合球面 一条线

$Z_{Boundary}$ = 

Z_{Bulk}() $= -1$ Z_{Bulk}() =

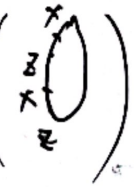
顶点.

因此: $(-1) \cdot Z_{Boundary}$() $\underset{gauge}{=} (+1)$ $Z_{Boundary}$()

$\Rightarrow$ Boundary is gauge non-invariant !

On Lattice ① • ⬭ ⬭ ... ⬭ • Non-trival. 有1a边界态

② ⬭ ⬭ ... ⬭ No 边界态

计算 $Z_{①Boundary}$ = Z()

$= (-1) \times \mathrm{Tr}_{\text{whole Hilbert space}}$ 即 $(-1)\times\chi(0)$

把Bulk-Boundary一起考虑. well-defined 又 under gauge transformation
而Bulk-Boundary分开仅考虑 Boundary 又在gauge transformation下
变化. 因此Boundary无法由一个0维的Hamiltonian来描述 从而
在AKLT中必呈现两相. (1)

Summary: A_1 A_2 $(H \to T O)$

Anyon condensation. (1) $\theta_a = 0 \quad \forall a \in A$

(2) $\theta_{a,b} = 0 \quad \forall a,b \in A$

(3) $\theta_{c,a} \neq 0 \quad \forall c \in \bar{A}, \exists a \in A$

Example: toric code: (1) smooth boundary $A: \{1, m\}$

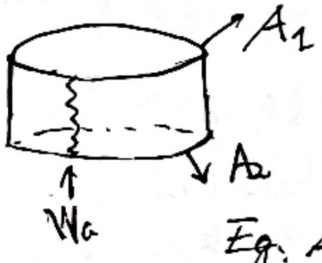
(2) rough boundary $A: \{1, e\}$

$\mathbb{Z}_4$ gauge theory: $\{e^i, m^j \mid i,j = 0, 1, 2, 3\}$

(a) $A: \{1, e, e^2, e^3\}$

(b) $A: \{1, m, m^2, m^3\}$

(c) $A: \{1, e^2, m^2, e^2m^2\}$.



能够连接上下边界的 W_a string.

$W_a: a \in \{A_1 \cap A_2\}$

GS Degeneracy: $|A_1 \cap A_2|$

Eg: $A_1: \{1, e\}$ $A_2: \{1, m\}$ $A_1 \cap A_2 = \{1\} = 1$

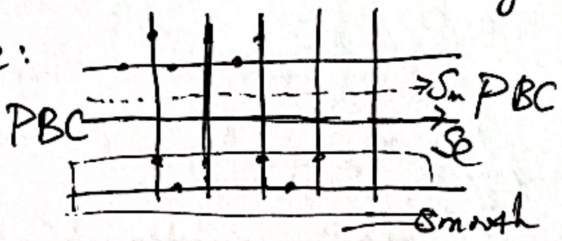
Eg: $A_1 = A_2$ toric code: GSD $= |A_1|^{\frac{\text{toric}}{\text{code}}} 2$

Corollary: If $\mathcal{C}$ has one and only one subgroup A the upper bound corresponds to $A \Rightarrow$ The whole system is either degenerate or gapless. 为 gapped boundary, 必 $A_1 = A_2$, $D = |A_1|$ = degene

I. similar 表述 as LSM theorem]

8. edge lattice model. rough boundary {1.e}

Toric code:



如同上页 ~~ebi~~ 圆柱. 存在 {1} unique ground state;

$S_e|\Psi\rangle = S_m|\Psi\rangle = |\Psi\rangle$

修改 Toric code Hamiltonian:

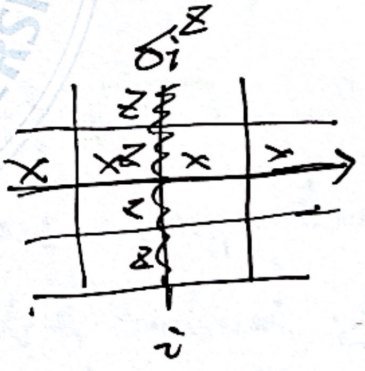
$$H = -\sum_p B_p - \sum_{v\notin \text{lower bound}} A_v - \lambda \sum_{v\in \text{lower bound}} A_v$$

consider $\lambda=0$, Fine-tuned GSD $= 2^L$ 每个 lower bound 上的 A_v 为1与为-1没有区别. 得到了 lower bound = Effective spin chain 为1与为-1不改变 H. Lower bound 仅Bp起作用.

GS: $|a_1=\pm1, a_2=\pm1 \cdots a_L=\pm1\rangle$

Lower bound a_{v_i}

$S_e|GS\rangle \xrightarrow{\text{把e string挪到上边界等价}} |GS\rangle$



$|GS: a_i=-1\rangle$ 作用1个 (σ_i^z) operator 只作用在了 a_i 上面, $a_i=-1 \to a_i=1$. 因此用 σ_i^z 表示 GS.

$$|a_1 \cdots a_L\rangle = \prod_i (\sigma_i^z)^{\frac{(1-a_i)}{2}} |\Psi_m\rangle$$

因此: $S_m|GS\rangle = S_m \prod_i (\sigma_i^z)^{\frac{(1-a_i)}{2}} |\Psi_m\rangle$

$S_m = (X_1 X_2 \cdots X_i \cdots)$ 每穿过一个 σ_i^z 出一个负号. 则.

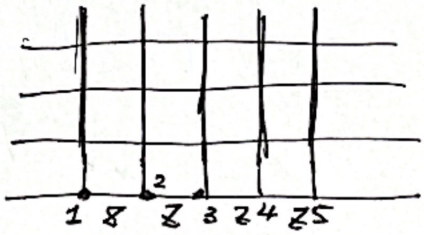
$$S_m|GS\rangle = \left(\prod_i a_i\right)|GS\rangle$$

$S_e \Leftrightarrow \hat{1}$ $S_m \Leftrightarrow \left(\prod_i a_i\right)$

问：哪些 string 中 operator is symmetric（对易）under S_m (± 1)

(Z_1) operator：$(\sigma_i^z \sigma_{i+1}^z)$ 让 S_m 穿2次 与 S_m 对易

$(A_v) = \sigma_i^x$ 自然与 S_m 对易



写出 minimal Hamiltonian 与 S_m 对易

$$H = -J\sum \sigma_i^z \cdot \sigma_{i+1}^z - h\sum_i \sigma_i^x$$

所谓与 S_m 对易用的，即阻止 lower edge 与 upper edge 的关联。σ_i^x 只发生下边界。$\sigma_i^z \sigma_{i+1}^z$ 为于下边界。仅一边界有关之 H，Transverse Ising model

H: No interaction between two edges.

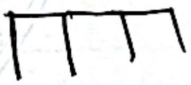
GSD = 什么

$J=0$ h 有限：$H = -\phi \sum_p B_p - \sum_v A_v \Rightarrow -h\sum_i \sigma_i^x$ 就是 A_v $= \hat{I}$

没有改变 H toric-code. 下边界仍为 m-condensed phase.

$h=0$ J 有限：$H = \cdots\cdots - J\sum \sigma_i^z \sigma_{i+1}^z$ 固定 $\sigma_i^z = \sigma_{i+1}^z = \cdots =$

$-\sum_p B_p$： 

$\overset{\text{edge}}{=} (Z_1 Z_2 Z_3 \times 1) =$ 

$-J\sigma_i^z \sigma_{i+1}^z$

如同{1.m} rough boundary：[e-condensed phase = rough boundary]

因此，在 Hedge 下，上下边界在 $h=0$ J 有限时都为 e-condensed phase.

GSD = 什么 $= 2$ 为 $[\uparrow\cdots\uparrow$ 与 $\downarrow\cdots\downarrow]$ 为 SSB. 即 h/J 调节为下 edge 中的 e-condensed phase 与 m-condensed phase 的竞争相图.

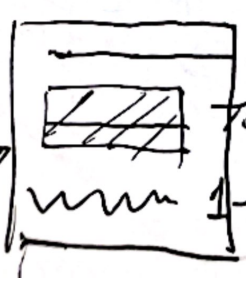
{|↑↑⟩, |↓↓⟩}

SSB — PM → h/J

0

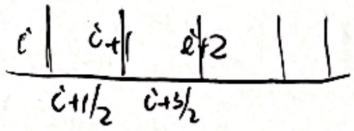
GSD = 2　GSD = 1

$|\rightarrow\cdots\rightarrow\rangle$

Model 

2维

$W=1$ limit: Comment



要满足 $B_p|\Psi\rangle=|\Psi\rangle$. 必要:

$Z_i Z_{i+1/2} Z_{i+1}|\Psi\rangle=|\Psi\rangle \ \forall i$

Apply e-string $S_e: \left(Z_{1/2} Z_{3/2} \cdots Z_{\frac{N+1}{2}}\right)|\Psi\rangle=|\Psi\rangle$

S_e: global $\mathbb{Z}_2$ symmetry: 同时反向.

但可以有 local 的 gauge invariance condition $Z_i Z_{i+1/2} Z_{i+1}|\Psi\rangle=|\Psi\rangle$

$S_e=\left(\prod_i Z_{i+1/2}\right)$ 的 gauge $Z_{i+1/2}' \to \left(Z_i Z_{i+1}\right) Z_{i+1/2}$.

ie: e-condensed $\Leftrightarrow$ gauge invariance ~~tion~~ condition.

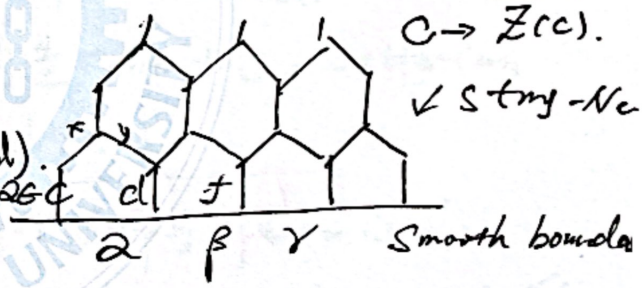
(? Kramers-Wannier duality)

String-Net 上. T. 放入 boundary

If $Z(C)\equiv 1$ has only one Lavage algebra.

+ one topological boundary (边界也是C. 为gapped)

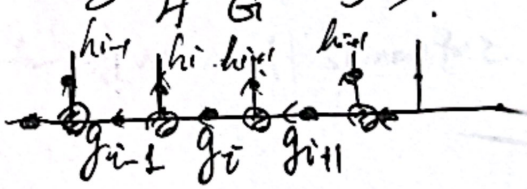
$\Rightarrow$ the other boundary cannot be trival.



$W=1$ limit: (double semion, $G=\mathbb{Z}_2$, $F=\nu_3\in \mathcal{H}^3(G, U(1))$).

每个 Link 有 spin, 放一个群元 $g_i\in G$

g 群元定义了 local operator.

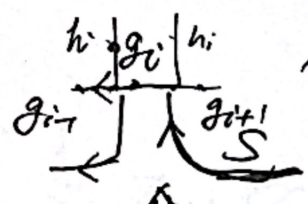


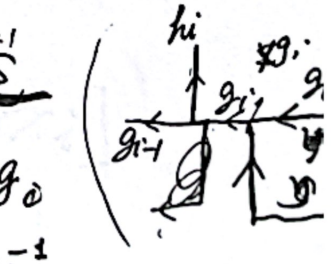
Satisfies $A_v|\Psi\rangle=|\Psi\rangle$ A_v

$g_{i-1}\otimes h_i \xrightarrow{\Pi} g_i$ 仅能有效仅 $A_v=1$

(Hamiltonian symmetry Requirements).

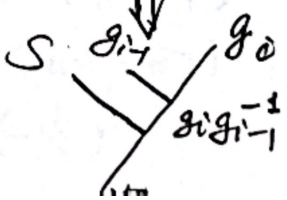
$h_i=g_i g_{i-1}^{-1}$



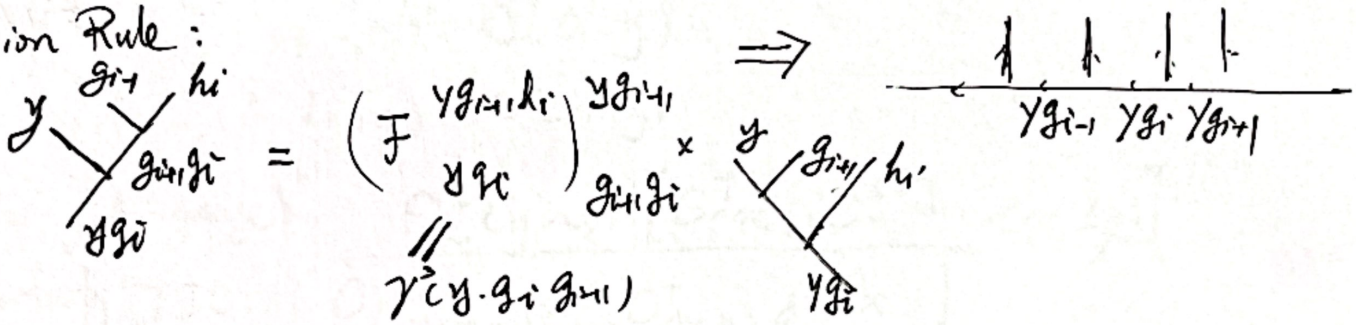


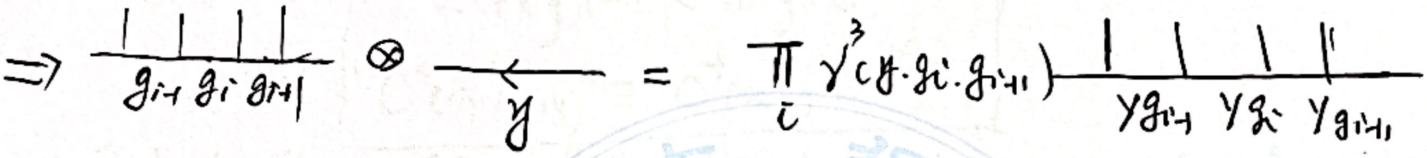
Apply String: (S) 在群的视角下: fusion.

$S\to S\otimes g_i$

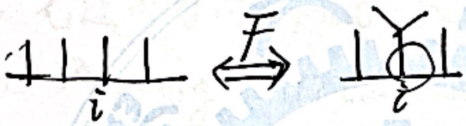


Fusion Rule:





如果 F move st:



在i处定义为可 local operator $\hat{O}_i \left| \begin{smallmatrix} & g_i \\ g_{i-1} & \; g_{i+1} \end{smallmatrix} \right\rangle = \underbrace{W_{g_i}}_{\text{number}} \left| \begin{smallmatrix} & g_i \\ g_{i-1} & \; g_{i+1} \end{smallmatrix} \right\rangle$

该 $\hat{O}_i$ 必有 $g_i \in G$ 对称有 G 对称性

满足. $\hat{O}_i \hat{U}_G = \hat{U}_G \hat{O}_i$ 因此可以构造 symmetric hamiltonian

$H = (\sum_i O_i)$ (Hamiltonian 算符一定是 local 的)

Remark. **What's a symmetry?** 虽然 $\hat{S}_a$ 可写作矩阵每 operator 运算

但 symmetry 必要 $\hat{S}_a \hat{X}_i \hat{S}_a^{-1} = \hat{X}_i$ 不改变 operator 的 locality 性质

Example: $G = \mathbb{Z}_2$ $\nu^3 \in H^3(\mathbb{Z}_2, U(1))$

(1) $\nu^3 = 1$, toric code), (X_i) local operator 为 minimal symmetry.

operator. X_i 与上面一整 (g_i) 都对易. 因此

$$H_{\text{toric-code-minimal}} = \sum_i X_i$$

(2). $\nu_3(a,b,c) = (-1)^{abc}$ ($\mathbb{Z}_2$ SPT edge)

$$H = \tfrac{1}{2} \sum_i \left[(Z_{i-1} + Z_{i+1}) X_i + (1 - Z_{i-1} Z_{i+1}) \tilde{X}_i \right]$$

(Gapless phase SU(2), CFT)

(Luttinger Liquid $K = 1/2$)