

Flat Band Superconductivity

Part 2



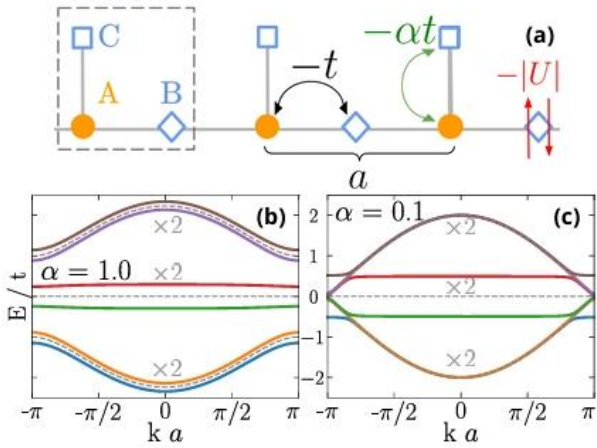
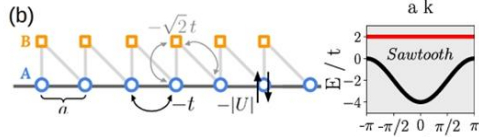


一维平带模型中的量子几何与超导关联文献分类

模型	Bipartite	Dispersive band	Uniform pairing	quantum metric / metric length	metric length 可调?	主要结论
Stub lattice / stub chain	✓	✓	×	$\langle g \rangle$ 由 hopping 参数 α 控制; $\ell_{qm} = \sqrt{\langle g \rangle}$	✓	典型的 1D 可调 quantum metric 模型; D_s 与 $\langle g \rangle$ 的关系随 gap/耦合改变。
Sawtooth chain	×	✓	×	固定 minimal QM; $\ell_{qm} \approx \sqrt{\langle g_{min} \rangle}$	×	子晶格不等价, $\Delta_A \neq \Delta_B$; D_s 低 U 线性, 但不简单 $\propto$ QM。
Creutz ladder / Creutz lattice	×	×	✓	QM 固定; 由 winding / band geometry 决定	×	标准 all-flat-band 模型; 几何贡献最清晰, pair correlations 更稳健。
Lieb-Kitaev model (三带 Lieb-like p-wave TSC)	×	✓	✓	QML $\xi_{QM} = \int G(k) dk / 2\pi$; $V/J \rightarrow 0$ 时 $\xi_{QM} \approx (\sqrt{2}/4)(J/V)a$	✓	MZM 局域长度由 QML 控制; 长 QML 可导致两端 MZM 杂化和超长程 CAR。

Stub chain

[1]



$$h(k) = \begin{pmatrix} 0 & -t(1 + e^{-ik}) & \alpha t \\ -t(1 + e^{ik}) & 0 & 0 \\ \alpha t & 0 & 0 \end{pmatrix}$$

$$\mathcal{H}_{\text{BdG}}(k) = \begin{pmatrix} h(k) - \mu I_3 & \Delta I_3 \\ \Delta^* I_3 & -h^*(-k) + \mu I_3 \end{pmatrix}$$

$$E_{0,\pm}^{\text{BdG}}(k) = \pm \sqrt{\mu^2 + |\Delta|^2}$$

- Further thinking: with attractive U , the MF Δ is different among orbitals due to different bloch weight ω_α , with A weight zero, the flat band now is relevant to k , and the band dispersion now is a superposition of orbitals:

$$\Delta_\alpha = -U \langle c_{i\alpha\downarrow} c_{i\alpha\uparrow} \rangle.$$

$$E_2^\pm = \pm \left(\Delta_B \frac{\alpha^2}{d^2(k)} + \Delta_C \frac{f_x^2(k)}{d^2(k)} \right).$$

- These gaps can be obtained by carrying out self-consistent solution of multi-orbitals to first order at small U , therefore, calculate min(E) with k , one can get the quasi-particle Gap: $\alpha_c = 1.155$

$$\Delta_B = \frac{|U|\alpha}{2\sqrt{4 + \alpha^2}},$$

$$\Delta_C + \Delta_B = \frac{|U|}{2}.$$

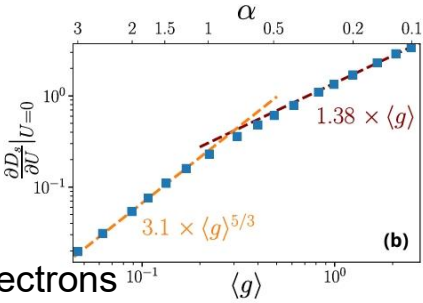
$$\Delta_{QP} = \begin{cases} \Delta_B & \text{for } \alpha \leq \alpha_c \\ \frac{\alpha^2 \Delta_B + 4\Delta_C}{\alpha^2 + 4} & \text{for } \alpha \geq \alpha_c, \end{cases} \quad \begin{matrix} k = \pi \\ k = 0 \end{matrix}$$

- Superfuldty weight can also be calculated , which also consists of two orbitals.

$$D_s = -\frac{1}{4}\alpha^2 \Delta_B (F(\eta) + \eta F'(\eta)) + \frac{1}{2}\Delta_C (F(\eta) + F'(\eta) - \eta F'(\eta)).$$

$$\alpha \ll 1 \quad D_s = \frac{3}{8\alpha}|U| = \frac{3}{2}|U|\langle g \rangle$$

$$\alpha \gg 1 \quad D_s = \frac{3}{\alpha^4}|U| = 12|U|\langle g \rangle^2$$



- For small α , A-B dispersion is strong , the compact localized state shows FB-electrons are mainly located at C, which shows the Wannier spreads between unit cells, metric mainly provided by C is just ordinary situation.

$$|\text{CLS}_i\rangle = \frac{1}{\sqrt{2+\alpha^2}}(|C_i\rangle + |C_{i+1}\rangle - \alpha|B_i\rangle).$$

$$w_C \sim O(1), w_B \sim O(\alpha^2).$$

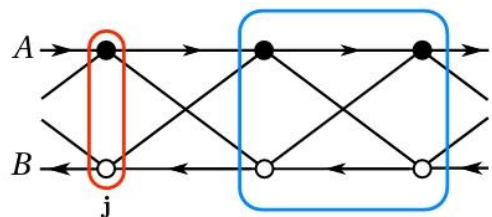
$$g(k) = \langle \partial_k \psi_k^{FB} | \partial_k \psi_k^{FB} \rangle - |\langle \psi_k^{FB} | \partial_k \psi_k^{FB} \rangle|^2. \quad \langle g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk g(k).$$

$$g(k) = \frac{\alpha^2 \sin^2 \frac{k}{2}}{(\alpha^2 + 4 \cos^2 \frac{k}{2})^2} \quad \left| \quad \langle g \rangle = \frac{1}{2|\alpha|\sqrt{\alpha^2 + 4}} \right|$$

- For big α , A-C dispersion is strong, FB-electrons are mainly located at B, but B has a very localized Wannier, B doesn't provide SW ! Still we should rely on C, but now FB-C-electron has $w_C \sim \frac{1}{\alpha^2}$,i.e . FB-Gap-C = Ug , while $D_s \sim \Delta_C \langle g \rangle$. , so we have the square relation.
- $$w_B \approx 1, w_C \sim \frac{1}{\alpha^2}.$$

Cruetz chain

[2][3] How to prove a superconducting phase in 1-D flat band system ?



$$\mathcal{H} = -it \sum_{j,\sigma} \left(c_{j,\sigma}^{A\dagger} c_{j+1,\sigma}^A - c_{j,\sigma}^{B\dagger} c_{j+1,\sigma}^B + \text{H.c.} \right) - t \sum_{j,\sigma} \left(c_{j,\sigma}^{A\dagger} c_{j+1,\sigma}^B + c_{j,\sigma}^{B\dagger} c_{j+1,\sigma}^A + \text{H.c.} \right) + U \sum_{j,\alpha} n_{j,\uparrow}^\alpha n_{j,\downarrow}^\alpha$$

- We should start with the definition of SW by (Torma 2015), and we know at zero temperature with fixed particle number, grand potential can substitute to G.S energy,

$$[D_s]_{i,j} = \frac{1}{V\hbar^2} \frac{\partial^2 \Omega}{\partial q_i \partial q_j} \Big|_{\Delta_i, \mu, q=0}$$

$$D_s = \pi L \frac{\partial^2 E_0(\Phi)}{\partial \Phi^2} \Big|_{\Phi=0}$$

- $q = \phi = \frac{\Phi}{L}$ The original SW shows stiffness with given scalar field, result in $t(q)$ with a phase factor of q , now the phase factor provided by the field becomes, we use it to calculate G.S energy

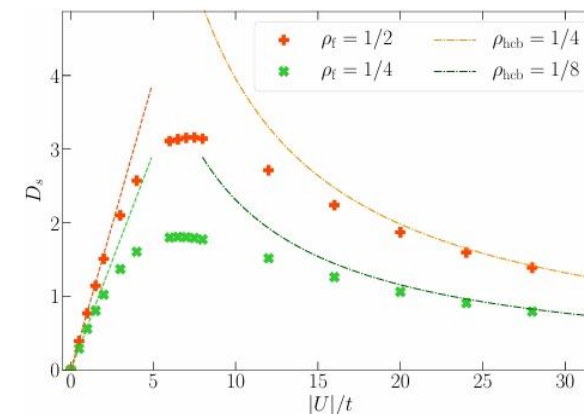
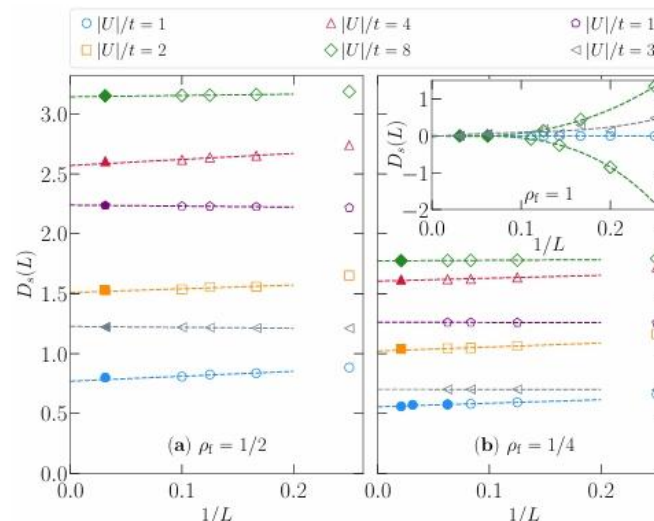
$$c_j^\alpha \rightarrow e^{i\phi j} c_j^\alpha,$$

$$\mathcal{H}(\Phi) = -it \sum_{j,\sigma} \left[e^{i\phi} c_{j,\sigma}^{A\dagger} c_{j+1,\sigma}^A - e^{i\phi} c_{j,\sigma}^{B\dagger} c_{j+1,\sigma}^B + \text{H.c.} \right] - t \sum_{j,\sigma} \left[e^{i\phi} c_{j,\sigma}^{A\dagger} c_{j+1,\sigma}^B + e^{i\phi} c_{j,\sigma}^{B\dagger} c_{j+1,\sigma}^A + \text{H.c.} \right] + U \sum_{j,\alpha} n_{j,\uparrow}^\alpha n_{j,\downarrow}^\alpha, \quad \phi = \Phi/L.$$

- PS: This method can define the many body quantum metric, but due to difficulty to find many body ground state, articles shows numerical results only for ED at small sizes. (Still can try) [7][8]

- Choose second derivation first, then extrapolate. where ρ is the filling of lower flat band, theory predicts in weak coupling:

$$D_s = \pi |U| \rho (1 - \rho). \quad D_s(L) \approx \pi L \frac{E_0(L, \Delta\Phi) + E_0(L, -\Delta\Phi) - 2E_0(L, 0)}{(\Delta\Phi)^2}.$$



- True SW should be invariant to finite size, when apply a flux ϕ on each site, for non-half-filling, we have gapless pair-fluid, (pair Luttinger) energy cost by the flux is $\Delta E_L(\Phi) \sim L \phi^2 = \frac{1}{L} \Phi^2$. so we fit $D_s(L) = D_s(\infty) + \frac{a}{L} + \frac{b}{L^2} + \dots$, For finite $D_s(\infty)$, we set $D_s(L) = D_s(\infty) + \frac{b}{L}$. For the half-filling case, we have zero $D_s(\infty)$, more like a band insulator, 1D gapped system has $e^{-r/\xi}$ response to local disturbance. i.e. flux at the boundary, so with finite effect of $e^{-L/\xi}$, we fit $D_s(L) \sim A e^{-L/\xi}$ eff.

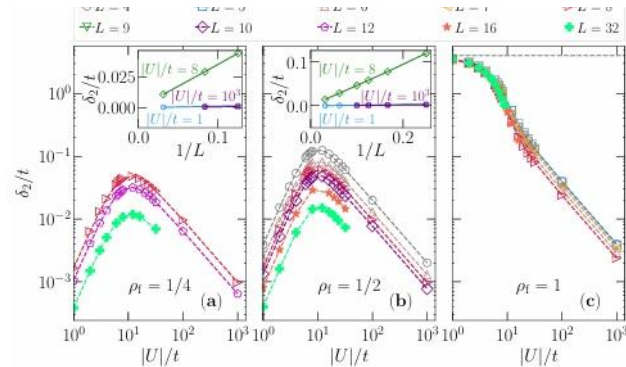
For weak coupling, ordinary flat band $D_s = \pi |U| \rho (1 - \rho)$.

For strong coupling, pair prefers the same site, $b_j^\dagger |0\rangle = |\uparrow\downarrow\rangle$, behaves like hard core bosons, with second order pairing process $J_{\text{pair}} \sim \frac{2t^2}{|U|}$.

Cruetz chain



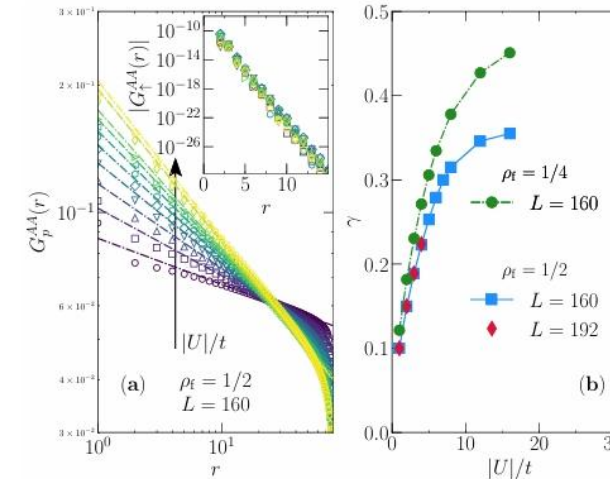
$$\delta_m \equiv \frac{1}{m} [E_0(N+m) + E_0(N-m) - 2E_0(N)],$$



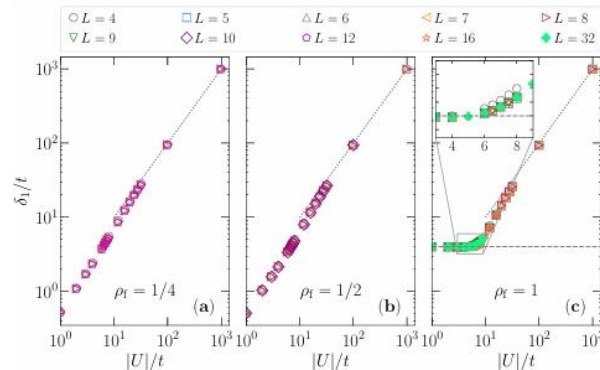
$$G_p^{\alpha\beta}(r) = \langle \Delta_{j+r}^{\alpha\dagger} \Delta_j^\beta \rangle,$$

$$G_\sigma^{\alpha\beta}(r) = \langle c_{j+r,\sigma}^{\alpha\dagger} c_{j,\sigma}^\beta \rangle,$$

$$\Delta_j^\alpha \equiv c_{j,\uparrow}^\alpha c_{j,\downarrow}^\alpha,$$



- Pair Gap (m=2): non-half-filling is pair Luttinger, pair gap should be gapless, half-filling is pair band insulator, pair gap should be gapped



- Single particle Gap (m=1): half-filling the lower band is completely filled and the cost in energy to add an extra particle is then 4t, strong coupling the gaps are proportional to U, single particle excitation breaks cooper pair, costing pair energy U.

- Pair correlation functions: the power-law decay of pair green function is the result of non-half-filled pair Luttinger
- Single particle function: the low-energy object is not an individual fermion, but a Cooper pair. Extracting a single fermion requires breaking a pair, and therefore costs a finite energy gap. A single-particle Green function associated with a gapped excitation decays exponentially.
- These results show if single function decays exponentially, the single-particle excitation is gapped, so the system is not an ordinary metal. If pair function decays algebraically, the pairs still have gapless coherent motion, These indicates the existence of **paired superfluid**

Drude Weight:
Superfluid Weight:

[4]

$$D = \lim_{L \rightarrow \infty} \pi L \frac{\partial^2 E_0(L, \Phi)}{\partial \Phi^2} \Big|_{\Phi=0}.$$

$$F_L(\Phi) = L[E_0(L, \Phi) - E_0(L, 0)], \quad F_\infty(\Phi) = \lim_{L \rightarrow \infty} F_L(\Phi),$$

$$D_s = \pi \frac{\partial^2 F_\infty(\Phi)}{\partial \Phi^2} \Big|_{\Phi=0}.$$



- Drude Weight: threading a flux Φ over a ring, if charge can move around the ring, the ground-state energy has non-zero curvature of Φ ; if charge is localized, the energy is almost insensitive to Φ . The curvature therefore measures the **charge stiffness** of Φ , i.e. whether or not charge is easy to move. It is a transport quantity. A non-superconducting ideal-metal can also have the curvature.(zero resistance)
- After taking the thermodynamic limit first, the system can be viewed as a continuum limit. The boundary twist can then be interpreted as a uniform phase gradient. The energy cost of this phase gradient is: $\Delta F \sim D_s (\nabla \theta)^2$. And $\nabla \theta = \frac{\Phi}{L}$. Therefore, the curvature now has the meaning of **phase stiffness**, and that is related to Meissner effect.
- Relationship to response theory: for $U=0$, define: $H(\phi) = H_0 - \phi J + \frac{\phi^2}{2} K + O(\phi^3)$.

$$\begin{aligned} \mathcal{H}(\Phi) = & -it \sum_{j,\sigma} [e^{i\phi} c_{j,\sigma}^{A\dagger} c_{j+1,\sigma}^A - e^{i\phi} c_{j,\sigma}^{B\dagger} c_{j+1,\sigma}^B + \text{H.c.}] \\ & -t \sum_{j,\sigma} [e^{i\phi} c_{j,\sigma}^{A\dagger} c_{j+1,\sigma}^B + e^{i\phi} c_{j,\sigma}^{B\dagger} c_{j+1,\sigma}^A + \text{H.c.}] \\ & + U \sum_j n_{j,\uparrow}^\alpha n_{j,\downarrow}^\alpha, \quad \phi = \Phi/L. \end{aligned}$$

- $\Lambda(q, \omega)$ is not analytical at $(0,0)$, so the order towards zero matters.
- Meissner effects asks whether Under a static transverse vector potential($\omega=0$), does the system generate a current in the continuum limit ($q \rightarrow 0$)? It is a static response under the continuum limit.
- Zero resistance asks whether Under a spatially uniform electric field($q=0$), can the system sustain a current with no decay in the long time limit($\omega \rightarrow 0$)? It is a dynamical transport quantity in the static limit.
- In high dimension, $q(q_x, q_y)$, we can set the vector potential in the x direction with $A_x(q_x, q_y \rightarrow 0)$. Drude weight is a uniform vector potential along x and asks whether the system supports a dissipationless current. Superfluid weight is the vector potential to along x, but let it vary spatially along y, and in one dimensional is all along x direction, behaving as a flux along the chain, these two are the same.
- However, that means in 1-D, $D \neq 0$ doesn't necessarily mean SC, but possibly be ideal metal.
- However, for flat band systems, no transport dispersion, no Drude weight. The finite D is provided by U and therefore can only due to phase stiffness. But we still can't precisely argue a superconducting phase, so we should still calculate pair correlation.

$$\begin{aligned} J &= -\frac{\partial \mathcal{H}}{\partial \phi} \Big|_{\phi=0} = it \sum_j c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \quad \langle J \rangle: \text{current} \\ K &= \frac{\partial^2 \mathcal{H}}{\partial \phi^2} \Big|_{\phi=0} = t \sum_j c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \quad \langle K \rangle: E_K \text{ positive} \end{aligned}$$

perturbation theory gives:

$$|1\rangle: |0(\phi)\rangle = |0\rangle + \phi \sum_{n \neq 0} |n\rangle \frac{\langle n|J|0\rangle}{E_0 - E_n} + O(\phi^2)$$

$$J(\phi) = -\frac{\partial \mathcal{H}(\phi)}{\partial \phi} = J - \phi K + O(\phi^2)$$

$$\langle J(\phi) \rangle = -\phi \left[\langle K \rangle - 2 \sum_{n \neq 0} \frac{\langle n|J|0\rangle^2}{E_n - E_0} \right] + O(\phi^2)$$

Linear-Response Gives:

$$\chi = \langle K \rangle - 2 \sum_{n \neq 0} \frac{\langle n|J|0\rangle^2}{E_n - E_0}$$

$$\langle J(\phi) \rangle = -\phi \chi \sim j = -\chi A$$

Hellman-Feynman Gives:

$$\frac{\partial E_0}{\partial \phi} = \langle 0 | \frac{\partial \mathcal{H}}{\partial \phi} | 0 \rangle \Rightarrow \frac{\partial E_0}{\partial \phi} \Big|_{\phi=0} = \langle K \rangle - \sum_{n \neq 0} \frac{\langle n|J|0\rangle^2}{E_n - E_0}$$

Therefore:

$$\frac{\partial^2 E_0(\phi)}{\partial \phi^2} \Big|_{\phi=0} = \underbrace{\langle K \rangle}_{\text{mean-}E_K} - \underbrace{\sum_{n \neq 0} \frac{\langle n|J|0\rangle^2}{E_n - E_0}}_{\text{current-current correlation function}} = \chi$$

$$\langle J(q, \omega) \rangle = A(q, \omega) A(q, \omega)$$

FBSC “coherence length”

[5][6]

$$G_{\lambda\eta}(\mathbf{r}) = \langle \hat{c}_{i\lambda,\sigma}^\dagger \hat{c}_{j\eta,\sigma} \rangle,$$

$$K_{\lambda\eta}(\mathbf{r}) = \langle \hat{c}_{i\lambda,\uparrow} \hat{c}_{j\eta,\downarrow} \rangle,$$



- For BCS-SC, cooper pair is formed by electrons within a shell near the Fermi surface, with velocity v_F , the pairing gap Δ sets a coherence time scale $1/\Delta$. Multiplying the two gives the spatial size of the cooper pair.

$$\xi_{\text{BCS}} \sim \frac{\hbar v_F}{\Delta}.$$

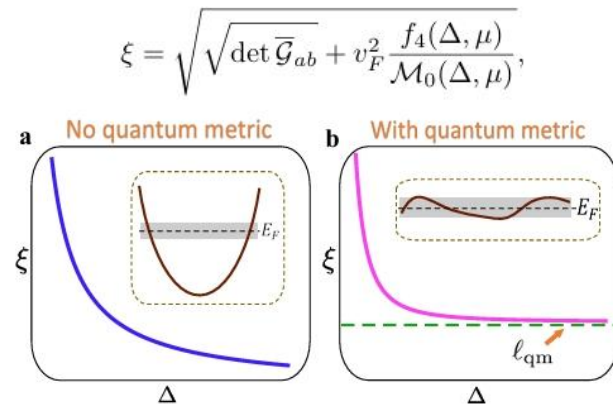
- We consider the pair correlation function:

$$C(\mathbf{r}, \mathbf{r}') = \sum_{\alpha} \langle a_{\alpha\uparrow}(\mathbf{r}) a_{\alpha\downarrow}(\mathbf{r}) a_{\alpha\downarrow}^\dagger(\mathbf{r}') a_{\alpha\uparrow}^\dagger(\mathbf{r}') \rangle_c.$$

- In a multi-flat band model, when projecting to low-energy energy band :

$$C(\mathbf{r}_{ij}) = \frac{T}{N} \sum_{\mathbf{q}} e^{-i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \sum_{nmhd} \sum_{\alpha, \mathbf{k}} u_{n\alpha}^*(\mathbf{k} + \mathbf{q}) u_{m\alpha}(\mathbf{k}) u_{h\alpha}^*(\mathbf{k}') u_{d\alpha}(\mathbf{k}' + \mathbf{q}) \langle c_{n\downarrow}(\mathbf{k} + \mathbf{q}) c_{m\uparrow}(-\mathbf{k}) c_{h\uparrow}^\dagger(-\mathbf{k}') c_{d\downarrow}^\dagger(\mathbf{k}' + \mathbf{q}) \rangle_c.$$

- The Bloch overlap is the character of multi-band systems, calculate the function with Wick contraction, using known Green function and that eventually results in quantum metric and BCS-length:

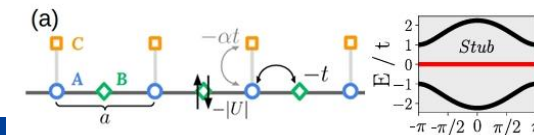
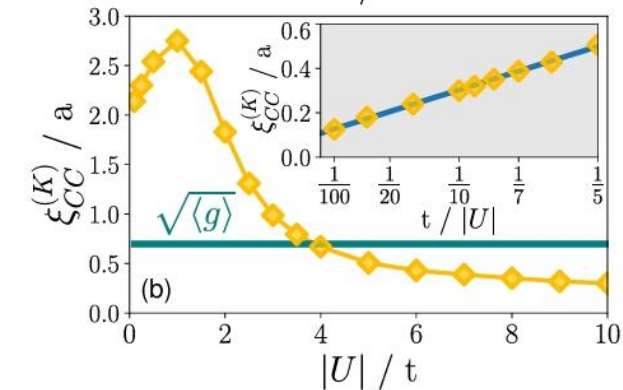


- Due to Wick contraction, it throws away background $|\langle \Delta \rangle|^2$, and green function behaves like single particle function, so the pair spread measures the correlation inside the pair, not the pair-pair correlation, so with exponential decay.

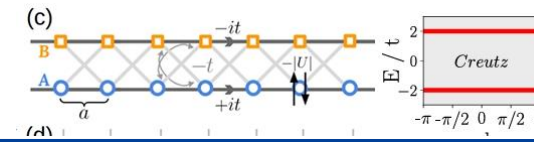
- However, another work argues the true length is way below metric length in FBSC, which from my perspective is a problem of definition, but gives me insight and comparison when numerical calculating correlation functions. Is pair spread the same with correlation length?

➤ My argument :

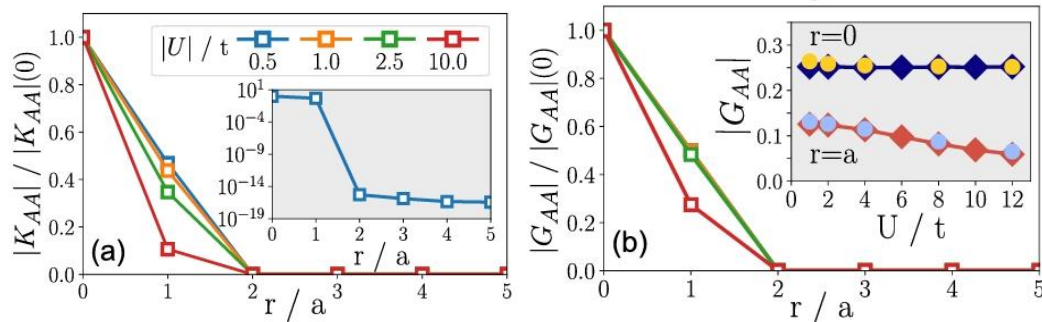
- The result of stub chain, they give BDG, with half-filled FB, upper FB empty, lower FB full, when $U=0$, no SW, but can have this pair correlation.
- The max and decay is explained by strong coupling behavior like BCS length onsite-binding, competing with the tail provided by SW.



FBSC “coherence length”



- The result of Creutz chain, when HF-FB, it doesn't have dispersive bands, and is band insulator in this case. So green function should like delta functions. Now the picture below they choose quarter filling, sites other than 0 and a shows result of zero within the numerical accuracy.



$$H_{\text{BdG}} = \sum_k \Psi_k^\dagger \begin{pmatrix} h_k^\dagger & \Delta_{I_{2 \times 2}} \\ \Delta_{I_{2 \times 2}}^* & -h_{-k}^\dagger \end{pmatrix} \Psi_k.$$

$$h_k^\dagger = \begin{pmatrix} -2t \sin(ka) - \tilde{\mu} & -2t \cos(ka) \\ -2t \cos(ka) & 2t \sin(ka) - \tilde{\mu} \end{pmatrix}.$$

$$\text{Quarter: } \tilde{\mu} = \mu + \frac{|U|}{4}.$$

Self-consistent solution:

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}} (|\Psi_2\rangle \pm |\Psi_3\rangle), \quad \langle c_{Ak\uparrow}^\dagger c_{Ak\uparrow} \rangle = \frac{1}{4} (1 + \sin(ka)).$$

$$E_\pm = \pm U/4.$$

$$G_{AA}(r) = \frac{1}{4} \delta_{r,0} - \frac{i}{8} \delta_{r,a} + \frac{i}{8} \delta_{r,-a}.$$

- The result for a half-filled chain, the use the pair spread defined by the previous work, and they in half-filled one-dimensional chain, it becomes:

$$\varepsilon_k = -2t \cos(ka), \quad \tilde{\xi}^2 = \sum_r r^2 |G(r)|^2, \quad G(r) = \frac{1}{N_c} \sum_k \frac{1}{2} \left(1 - \frac{\varepsilon_k}{E_k} \right) e^{-ikr},$$

- In the strong coupling regime: $\frac{|\varepsilon_k|}{\Delta} \ll 1$

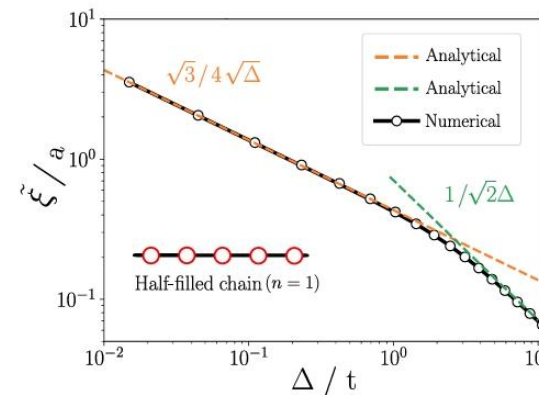
$$G(r) = \frac{1}{2} \delta_{r,0} + \frac{t}{2\Delta} (\delta_{r,-a} + \delta_{r,a}), \quad \tilde{\xi} = \frac{1}{\sqrt{2}} \frac{at}{\Delta}.$$

- In the weak coupling regime, expand to continuum limit, and in the limit of small Δ (K: Bessel function)

$$G(r) = \frac{1}{2} (\delta_{r,0} + I_1(r)), \quad I_1(r) = \frac{\Delta}{\pi t} \sin\left(\frac{\pi r}{2}\right) K_1\left(\frac{\Delta r}{2t}\right),$$

$$\sum_r r^2 |G(r)|^2 = \frac{2}{\pi^2} \frac{t}{\Delta} \int_0^\infty u^2 K_1^2(u) du = \frac{3}{16} \frac{t}{\Delta}$$

$$\tilde{\xi} = \frac{\sqrt{3}}{4} \sqrt{\frac{t}{\Delta}} a.$$



- So in weak limit, pair spread is actually second variance of green function tail, is it true pair spread?

FIGURE 8. $\tilde{\xi}$ in the half-filled standard chain as a function of the superconducting gap Δ (open circles). The dashed lines correspond to analytical expressions in the weak and strong coupling regimes.



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