



Interacting Kitaev Chain

宣嘉乐

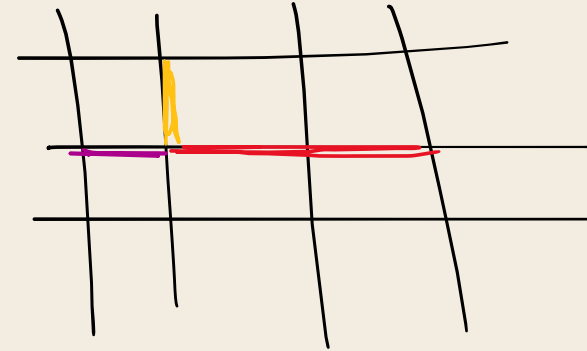
2025.10.13

ANNNI model and Interacting Kitaev model



➤ The ANNNI Model (2D) [1]

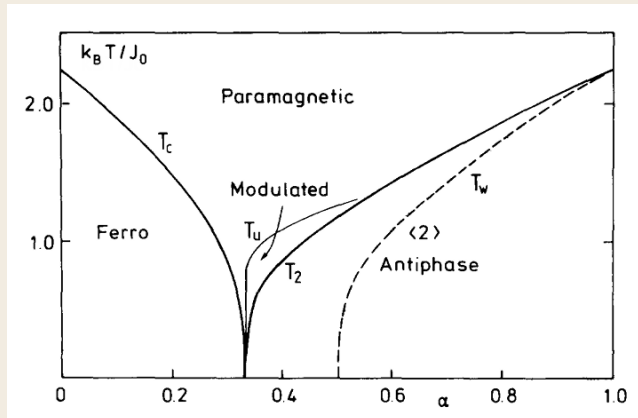
$$E[\{s\}] = - \sum_{i,j} (\tilde{J}_1 s_{i,j} s_{i+1,j} + \tilde{J}_2 s_{i,j} s_{i+2,j}) - \sum_{i,j} \tilde{J}_\perp s_{i,j} s_{i,j+1}$$



➤ Phase Diagram (Method : Classical MC)

When $K = -J_2/J_1 < 0$, The second nearest term only modifies the critical temperature.

When $K > 0$, occurs a multicritical point, where four phases meet , namely the ferromagnet , paramagnet , a floating phase , and the so-called “antiphase” .



➤ Example , following x axis :

++---++---++---++ <2> phase

++---++---++---++ <233> phase

“a floating phase” contains many these states



$$H = \left[\sum_{m,n} J_1 S_{m,n} S_{m,n+1} + J_2 S_{m,n} S_{m,n+2} + J_3 S_{m,n} S_{m+1,n} \right]$$

$$Z = \sum_{\{s\}} e^{-\beta H[\{s\}]}$$

$$= \sum_{\{s\}} \prod_{m(n)=1}^N \exp \left[\beta \sum_{n=1}^N J_1 S_n^2 S_{n+1}^2 + J_2 S_n^2 S_{n+2}^2 + J_3 S_n^2 S_{n+1}^2 \right]$$

把 m 视作时间切片, 内部为 $[e^{-\Delta t H}]^N$, 插入 $|\beta\rangle, |Q\rangle$ 路径积分.

$$|Q\rangle, |\beta\rangle: |S_{m,1}, \dots, S_{m,N}\rangle, |S_{m+1,1}, \dots, S_{m+1,N}\rangle$$

$$\text{定义: } \langle \beta | T | Q \rangle = \exp \left[\sum_{n=1}^N \beta J_1 S_n^2 S_{n+1}^2 + \beta J_2 S_n^2 S_{n+2}^2 + \beta J_3 S_n^2 S_{n+1}^2 \right]$$

$$\Rightarrow Z = \sum_{\{s\}} \langle 1 | T | 2 \rangle \langle 2 | T | 3 \rangle \dots \langle N-1 | T | N \rangle$$

$$= \text{Tr}[T^N] \quad T: e^{-\Delta t H}$$

$$\text{定义: } T = T_1 + T_2 + T_3$$

$$T_1 = \exp \sum_{n=1}^N \beta J_1 S_n^2 S_{n+1}^2 \stackrel{\times}{=} \langle \beta | \hat{T}_1 | Q \rangle$$

比如 $N=2$ $S_1^2 S_2^2$ 不会改变 $|Q\rangle$, 只有 $|\beta\rangle = |Q\rangle$ 才有值, 为矩阵对角元

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = T_1^Z T_2^Z \quad \text{Pauli-Z matrix}$$

$$\Rightarrow \hat{T}_1 = \exp [\beta J_1 \sum_n T_n^Z T_{n+1}^Z]$$

$$T_2 = \exp \sum_{n=1}^N \beta J_2 S_n^2 S_{n+2}^2 \stackrel{\times}{=} \langle \beta | \hat{T}_2 | Q \rangle$$

$$\text{与 } \hat{T}_1 \text{ 一样} = \exp [\beta J_2 \sum_n T_n^Z T_{n+2}^Z]$$



$$T_3 = \exp \sum_{n=1}^N \beta J_3 S_n^2 S_{n+1}^2 \stackrel{\times}{=} \langle \beta | \hat{T}_3 | Q \rangle$$

$$\Rightarrow \begin{matrix} N=1 & |1\rangle & |1\rangle \\ & 1 & -1 \end{matrix} \Rightarrow e^{\beta J_3 \hat{T}_3} = e^{\beta J_3 \hat{T}_3} + e^{-\beta J_3 \hat{T}_3}$$

$$= e^{2\beta J_3} \left(\frac{1}{2} + e^{-2\beta J_3} \hat{T}_3^x \right) \approx e^{2\beta J_3} \cdot e^{-2\beta J_3 \hat{T}_3^x}$$

$$\text{利用: } e^A = 1 + A \quad A \rightarrow 0, \text{ 更严谨为 } 1 + \delta^x \approx \exp(\arctan \delta^x)$$

$$\Rightarrow \hat{T}_3 = e^{2\beta J_3} \cdot e^{-2\beta J_3 \sum_i T_i^x}$$

N 足够大, 时间切片足够小, Trotter 分解成立:

$$\hat{T}_1 \hat{T}_2 \hat{T}_3 \approx e^{\hat{T}_1 + \hat{T}_2 + \hat{T}_3} = e^{-\Delta t H} \times \text{const}$$

$$\Rightarrow H = -J_1 \sum_{i=1}^{L-1} T_i^Z T_{i+1}^Z - J_2 \sum_{i=1}^{L-2} T_i^Z T_{i+2}^Z + B \sum_{i=1}^L T_i^x$$

$$\text{where: } B \propto \exp(-2\beta J_3)$$

然后做变换: ① 对偶变换, ② 旋转 (均么正, 保代数)

$$\textcircled{1} \begin{matrix} T_j^Z T_{j+1}^Z \rightarrow \delta_j^x \\ T_j^x \rightarrow \delta_j^Z \delta_{j+1}^Z \end{matrix} \quad \textcircled{2} \begin{matrix} \delta_j^x \rightarrow S_j^Z \\ \delta_j^Z \rightarrow (-1)^j S_j^x \end{matrix}$$

$$H'_0 = -J_1 \sum_j \delta_j^x + J_2 \sum_j \delta_j^x \delta_{j+1}^x + B \sum_j \delta_j^Z \delta_{j+1}^Z$$

$$H'_2 = -B \sum_j \delta_j^x \delta_{j+1}^x - J_1 \sum_j \delta_j^Z + J_2 \sum_j \delta_j^Z \delta_{j+1}^Z$$

$$\text{Def: } U = \frac{J_1}{B} K, \quad h = \frac{J_1}{B}, \quad K = -\frac{J_2}{J_1}$$

$$H(h, U) = - \sum_j \delta_j^x \delta_{j+1}^x - h \sum_j \delta_j^Z + U \sum_j \delta_j^Z \delta_{j+1}^Z$$

ANNNI model and Interacting

➤ Similarity with other models

When $U = 0$, The phase transition between these phases is described by the $c = 1/2$ Ising CFT, can be shown by entanglement entropy.

When $U = -1$ or 1 , $h = 0$, The phase transition between these phases is described by the $c = 1$ XY CFT, can be shown by entanglement entropy.

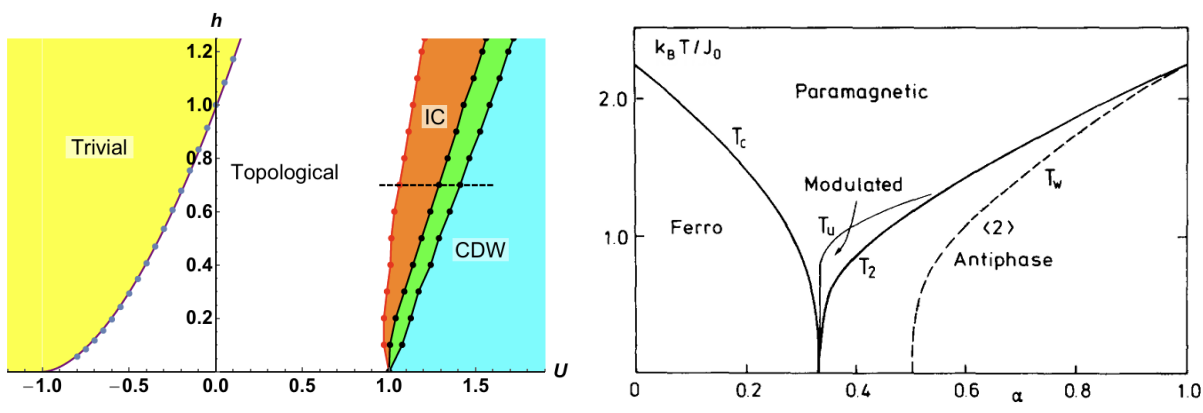


FIG. 1. The phase diagram for the model in Eq. (8). The green region, between the IC and CDW phase, is the esCDW phase.

$$U = \frac{J_1 K}{B}, h = \frac{J_1}{B},$$

$$B = \exp(-2\beta J_\perp), K = -J_2/J_1$$

$$U \Leftrightarrow J_2 \text{ x-axis remains}$$

$$K \Leftrightarrow h^{-1} \text{ 4-critical point}$$



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JW transformation:

最终 interacting Kitaev chain 定义在 spinless-fermion basis.

把 spin basis 做 JW transformation 到该表象.

$$\begin{cases} c_j^+ = c_j^+ e^{i\pi \sum_{i=1}^j n_i} \\ c_j^- = c_j^- e^{i\pi \sum_{i=1}^j n_i} \\ c_j^z = 1 - 2n_j \end{cases}$$

$$\text{第一项: } -\sum_{j=1}^{L-1} c_j^x c_{j+1}^x$$

$$= -\sum_{j=1}^{L-1} (c_j^+ e^{i\pi \sum_{i=1}^j n_i} + e^{i\pi \sum_{i=1}^j n_i} c_j^-) \cdot (c_{j+1}^+ e^{i\pi \sum_{i=1}^{j+1} n_i} + e^{i\pi \sum_{i=1}^{j+1} n_i} c_{j+1}^-)$$

$$= -\sum_{j=1}^{L-1} (c_j^+ - c_j^-) (c_{j+1}^+ + c_{j+1}^-)$$

$$\Rightarrow H(h, U) = -\sum_{j=1}^{L-1} (c_j^+ - c_j^-) (c_{j+1}^+ + c_{j+1}^-)$$

$$- h \sum_{j=1}^L (1 - 2n_j)$$

$$+ U \sum_{j=1}^L (1 - 2n_j) (1 - 2n_{j+1})$$

我们和 Kitaev chain 对比,

2-D 无外场 Ising $\rightarrow$ TFI-1D $\rightarrow$ Kitaev chain.

2-D ANNNI $\rightarrow$ "TFI"-1D $\rightarrow$ Kitaev Hubbard chain.

$U > 0$ case : Topological Phase

- Method : DMRG with conserved fermion parity P for different parameters
- **How DMRG works ?**
- Consider entanglement entropy , occupation number and energy

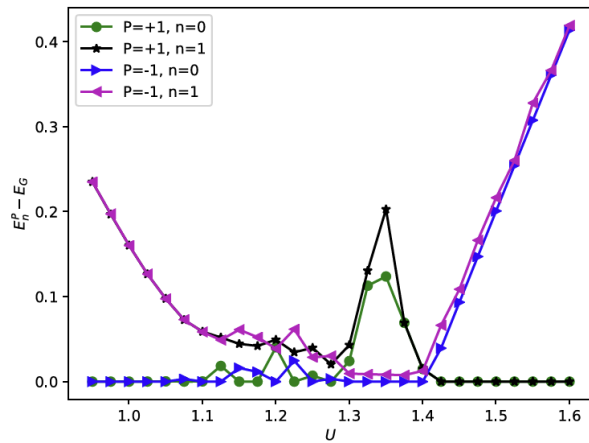


FIG. 4. Energy with respect to the ground state energy $E_n^P - E_G$ of the ground state ($n = 0$) and the first excited state ($n = 1$) in both parity sectors are plotted as a function of U for $h = 0.7$ and $L = 240$. Note that E_G is either E_0^{+1} or E_0^{-1} . The range of U corresponds to the dashed line in Fig. 1.

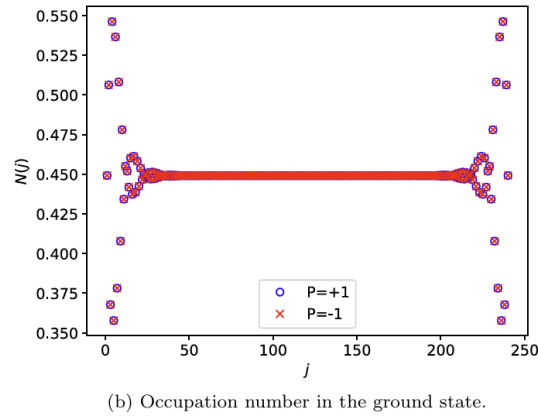
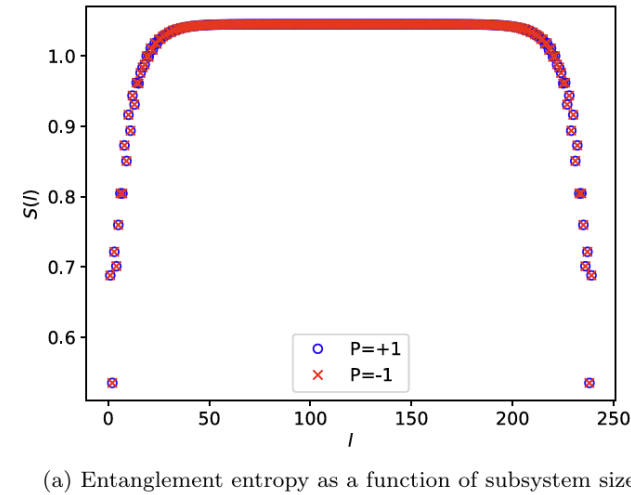


FIG. 5. Observables for $h = 0.7$ and $U = 0.95$, a point in the topological phase, are plotted for the two ground states. The system size is $L = 240$.



(a) Entanglement entropy as a function of subsystem size.

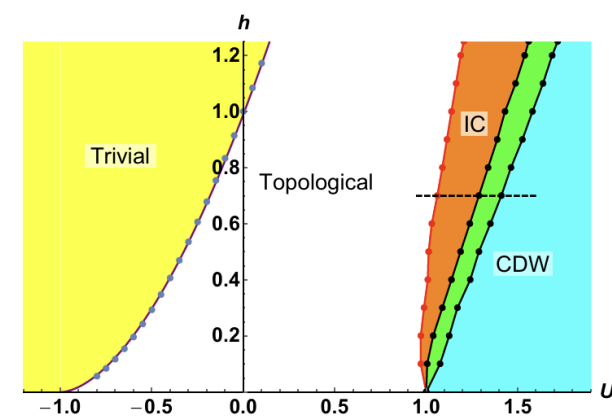


FIG. 1. The phase diagram for the model in Eq. (8). The green region, between the IC and CDW phase, is the esCDW phase.

➤ Properties of topological phase

The model is gapped and the ground state is doubly degenerate. The two ground states have different parity. The occupation number is constant in the bulk for both parity sectors.

$U > 0$ case : Incommensurate Phase

- Method : DMRG with conserved fermion parity P for different parameters
- Consider entanglement entropy , occupation number and energy

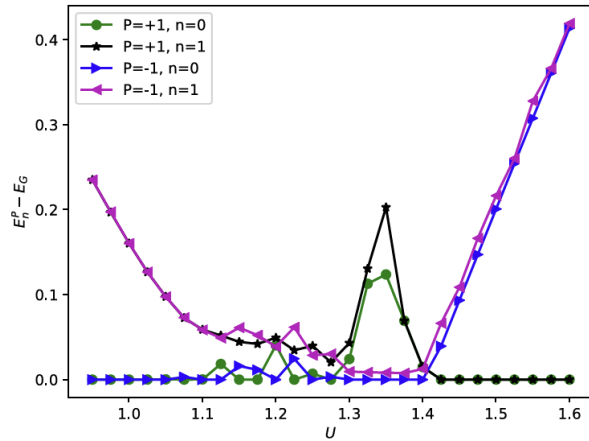
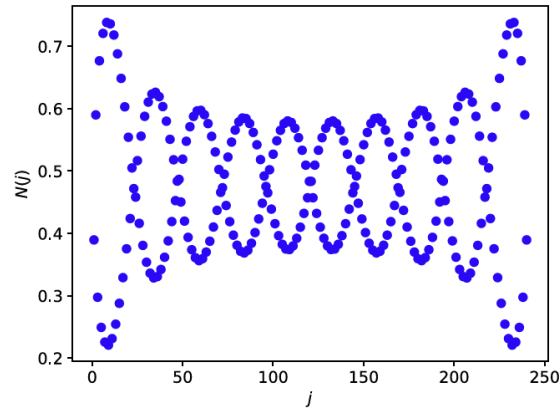
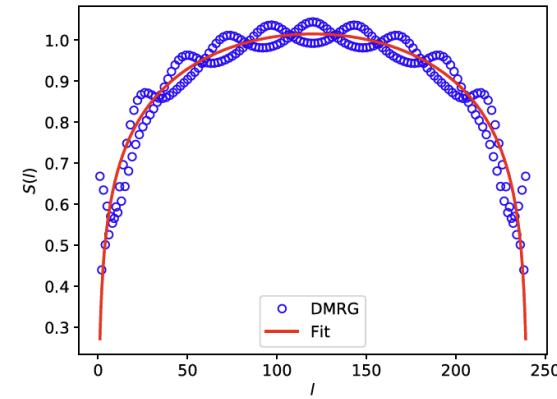


FIG. 4. Energy with respect to the ground state energy $E_n^P - E_G$ of the ground state ($n = 0$) and the first excited state ($n = 1$) in both parity sectors are plotted as a function of U for $h = 0.7$ and $L = 240$. Note that E_G is either E_0^{+1} or E_0^{-1} . The range of U corresponds to the dashed line in Fig. 1.



(b) Occupation number in the ground state.



(a) Entanglement entropy as a function of subsystem size.

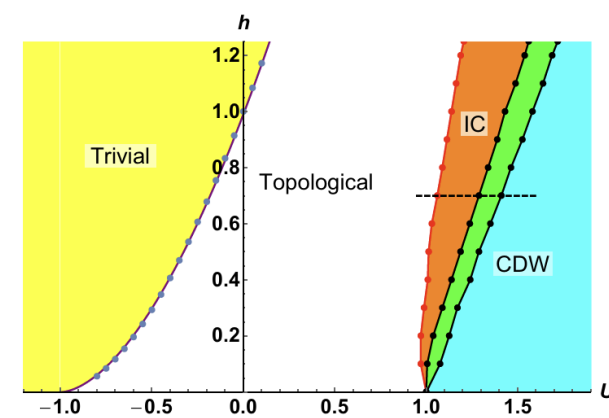
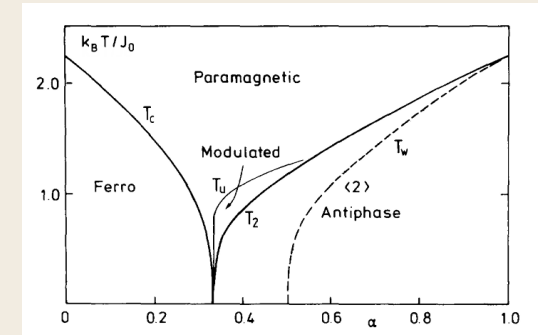


FIG. 1. The phase diagram for the model in Eq. (8). The green region, between the IC and CDW phase, is the esCDW phase.

- Example , following x axis :
 $++---++---++---++$ $\langle 2 \rangle$ phase
 $++-----++-----++$ $\langle 233 \rangle$ phase
 “a floating phase” contains many these states



➤ Properties of Incommensurate phase

The ground state is unique , calculating $c = 1.03$.

For a point in the topological phase close to the transition still $c < 1$, while in the IC phase one would obtain $c = 1$ also for larger system sizes .

Occupation number varies just like the “floating phase” .

Multiple crossings of the G.S. is because of the finite size effect.

$U > 0$ case : Incommensurate Phase

- Method : **DMRG** with conserved fermion parity P for different parameters
- Consider energy and energy shifting

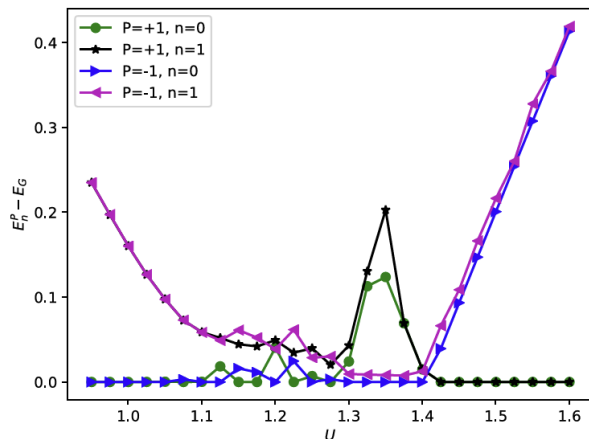
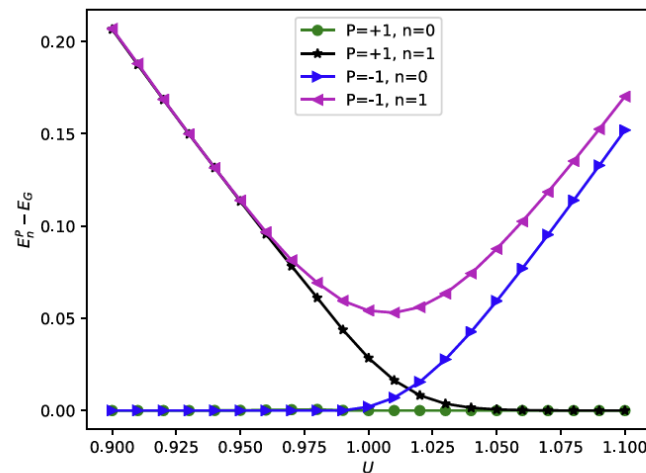
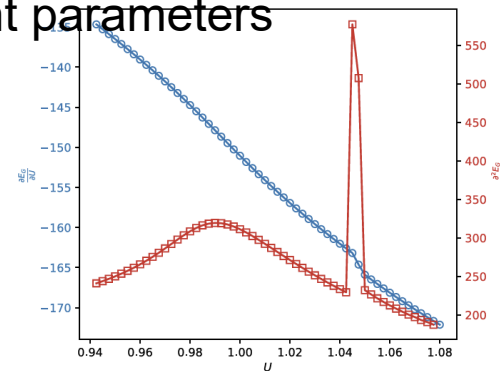


FIG. 4. Energy with respect to the ground state energy $E_n^P - E_G$ of the ground state ($n = 0$) and the first excited state ($n = 1$) in both parity sectors are plotted as a function of U for $h = 0.7$ and $L = 240$. Note that E_G is either E_0^{+1} or E_0^{-1} . The range of U corresponds to the dashed line in Fig. 1.



(a) Energy with respect to the ground state energy, $E_n^P - E_G$, of the ground state ($n = 0$) and the first excited state ($n = 1$) in both parity sectors are plotted as function of U for $h = 0.05$.



(b) The derivatives of the ground state energy, $\frac{\partial E_G}{\partial U}$ (the blue circles) and $-\frac{\partial^2 E_G}{\partial U^2}$ (the red squares) for $h = 0.1$ are plotted.

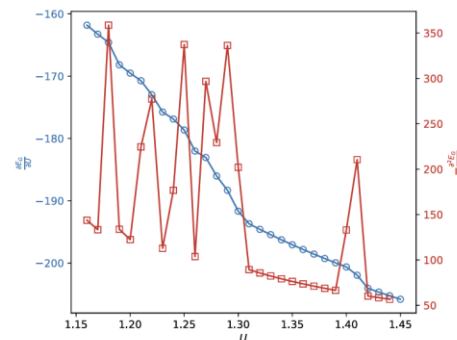


FIG. 7. The derivatives of the ground state energy $\frac{\partial E_G}{\partial U}$ (the blue circles) and $-\frac{\partial^2 E_G}{\partial U^2}$ (the red squares) are plotted against U for $h = 0.7$.

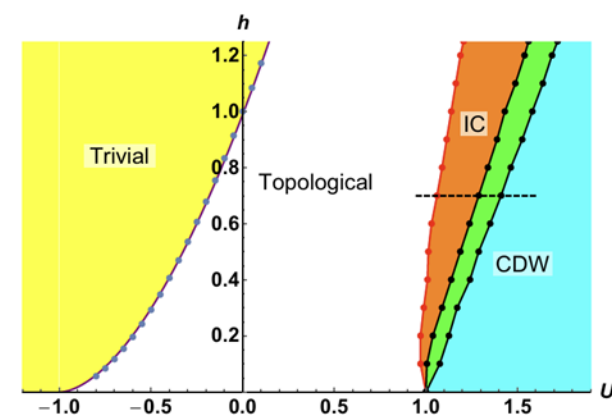
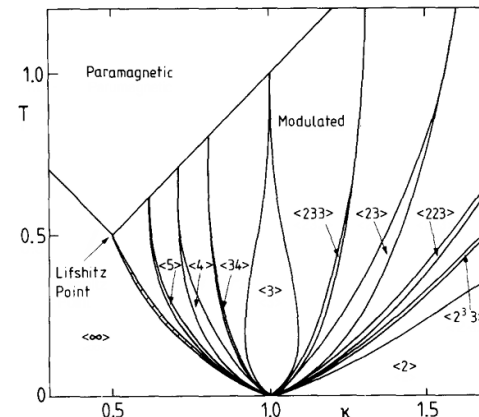


FIG. 1. The phase diagram for the model in Eq. (8). The green region, between the IC and CDW phase, is the esCDW phase.



➤ Properties of Incommensurate phase

G.S Energy discontinuity means first-order phase transition , more points for large h (small K) than small h (large K)

$U > 0$ case : CDW Phase

- Method : DMRG with conserved fermion parity P for different parameters
- Consider entanglement entropy , occupation number and energy

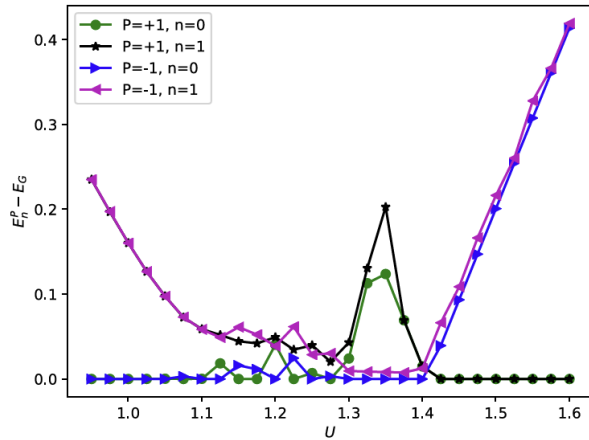
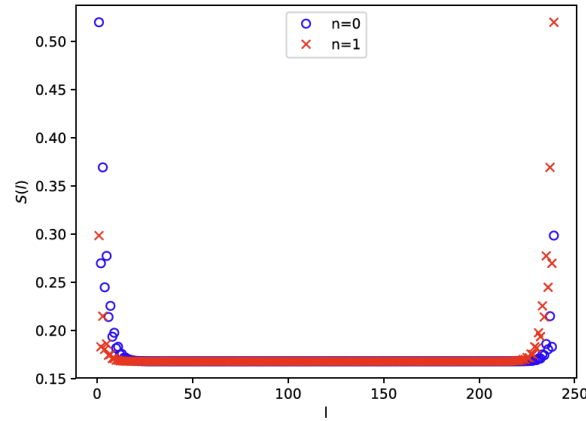


FIG. 4. Energy with respect to the ground state energy $E_n^P - E_G$ of the ground state ($n = 0$) and the first excited state ($n = 1$) in both parity sectors are plotted as a function of U for $h = 0.7$ and $L = 240$. Note that E_G is either E_0^{+1} or E_0^{-1} . The range of U corresponds to the dashed line in Fig. 1.



(a) Entanglement entropy as a function of subsystem size.

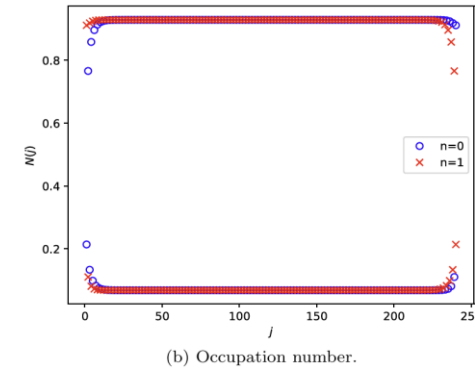


FIG. 14. Observables for the ground state ($n = 0$) and the first excited state ($n = 1$) at $h = 0.7$ and $U = 1.5$, a point in the CDW phase, are plotted. The system size is $L = 240$.

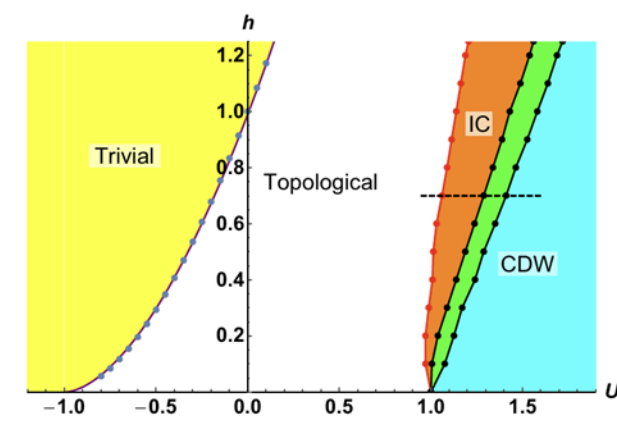


FIG. 1. The phase diagram for the model in Eq. (8). The green region, between the IC and CDW phase, is the esCDW phase.

	Even(1010...)	Odd(1010...)	Even(0101...)	Odd(0101...)
Np	L/2	(L+1)/2	L/2	(L-1)/2
L=4Z	Even parity		Even parity	
L=4Z+1	Odd parity		Even parity	
L=4Z+2			Odd parity	
L=4Z+3	Even parity		Odd parity	

➤ Properties of CDW phase

At large U limit, one can ignore the hopping and pairing terms , reaching the CDW phase.

The occupation number shows the pattern which one expects in the CDW phase, G.S is $(1010\cdots)$ and $(0101\cdots)$.

EE picture with a low value platform shows the phase is gapped with short-range correlations.

For $L = 4n$ (or $L = 4n + 2$), where n is a large integer, the two ground states belongs to the even (odd) sector. For L odd, however, one ground state has even parity, the other odd.

$U > 0$ case : Excited-state CDW phase

- Method : DMRG with conserved fermion parity P for different parameters
- Consider energy shifting

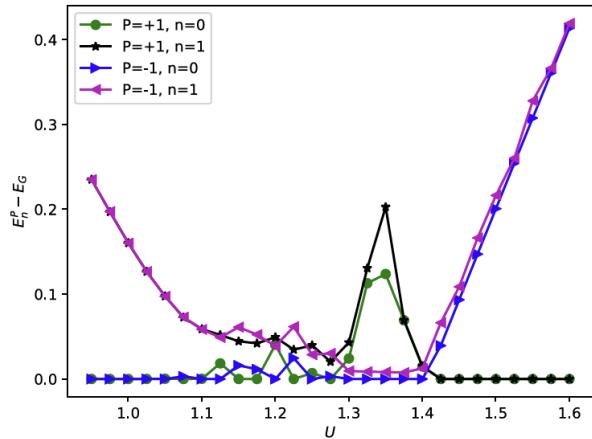


FIG. 4. Energy with respect to the ground state energy $E_n^P - E_G$ of the ground state ($n=0$) and the first excited state ($n=1$) in both parity sectors are plotted as a function of U for $h=0.7$ and $L=240$. Note that E_G is either E_0^{+1} or E_0^{-1} . The range of U corresponds to the dashed line in Fig. 1.

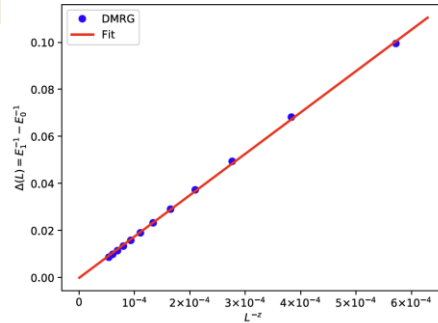


FIG. 10. Scaling of the gap $E_1^{-1} - E_0^{-1}$ for systems of the size $L=4n$ in the range of $L=64-240$ at $h=0.7$ and $U=1.325$.

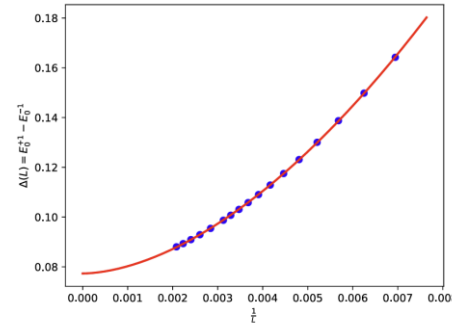
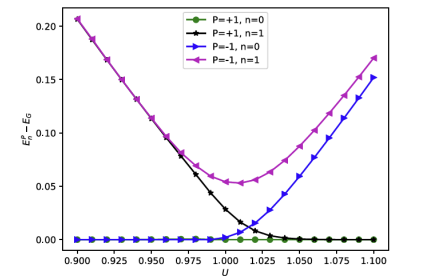


FIG. 11. Scaling of the gap $E_0^{+1} - E_0^{-1}$ for systems of the size $L=4n$ in the range of $L=144-480$ at $h=0.7$ and $U=1.325$. There is a finite gap $\Delta \simeq 0.077$ between the ground states of the two sectors.



(a) Energy with respect to the ground state energy, $E_n^P - E_G$, of the ground state ($n=0$) and the first excited state ($n=1$) in both parity sectors are plotted as function of U for $h=0.05$.

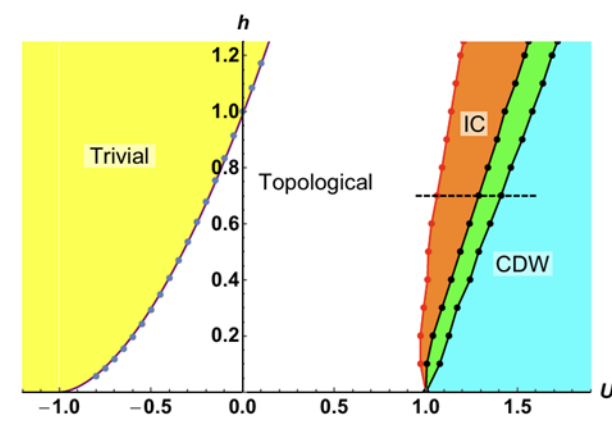
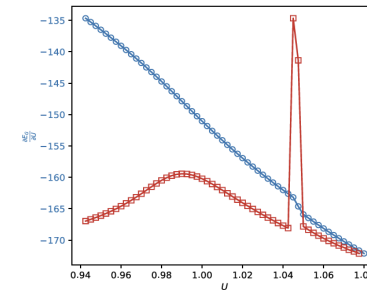


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(b) The derivatives of the ground state energy, $\frac{\partial E_G}{\partial U}$ (the blue circles) and $-\frac{\partial^2 E_G}{\partial U^2}$ (the red squares) are plotted against U for $h=0.1$.

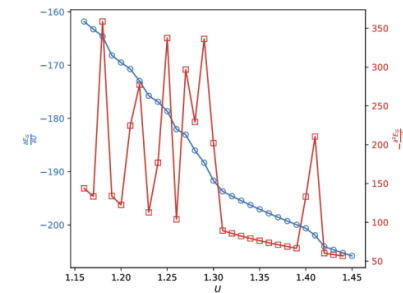


FIG. 7. The derivatives of the ground state energy, $\frac{\partial E_G}{\partial U}$ (the blue circles) and $-\frac{\partial^2 E_G}{\partial U^2}$ (the red squares) are plotted against U for $h=0.7$.

➤ Properties of Excited-state CDW phase

The model is gapless by looking at the first excited state in the odd sector, while there is a finite gap to the lowest energy state in the even sector. ($P=-1, n=1$), instead of ($P=1, n=0$), has the lowest excited energy.

Also G.S. parity is system size ($4Z, 4Z+1$) dependent.

There is a clear first-order transition for large h , while for small h two-kinds phase transition mix here, also hard to see. From the energy picture, some doubts here.

$U > 0$ case : Excited-state CDW phase

- Method : DMRG with conserved fermion parity P for different parameters
- Consider entanglement entropy , occupation number and energy

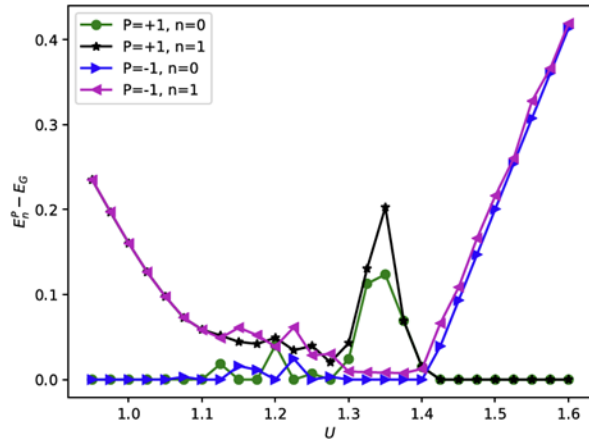


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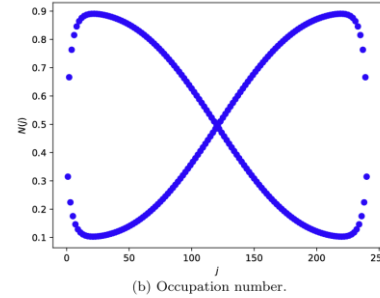
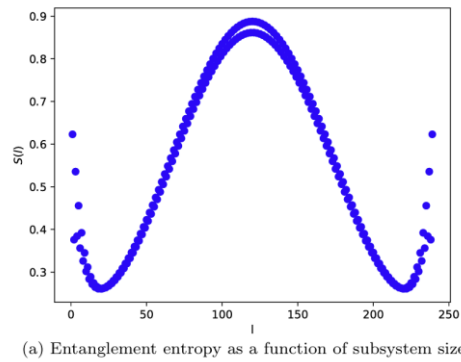


FIG. 9. Observables for the ground state at $h = 0.7$ and $U = 1.325$, a point in the esCDW phase are plotted. The system size is $L = 240$.

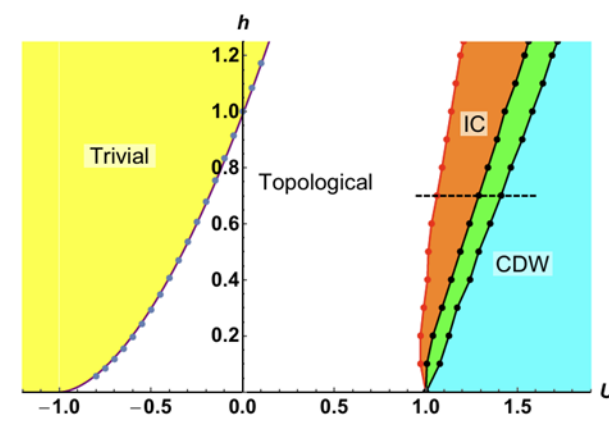
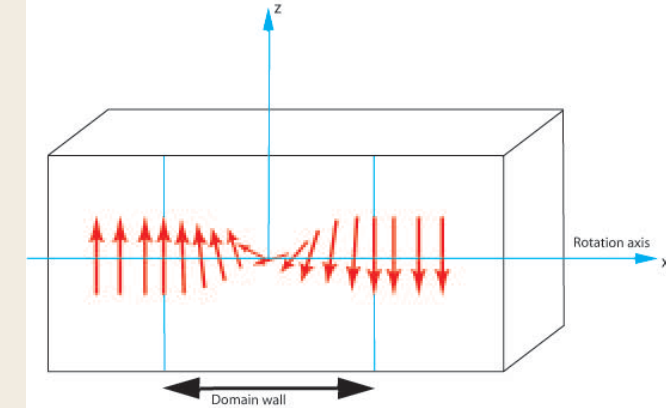


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➤ Properties of Excited-state CDW phase

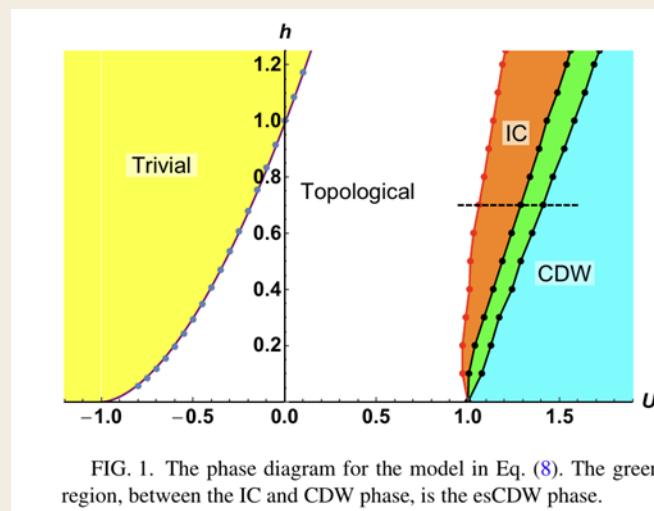
Occupation number and EE picture shows the CDW wave length is twice system size, with a π phase shift for neighboring sites.

➤ CDW excitation (孤子激发) e.g.(1010...11...1010) compare with “domain wall”

Left side is far from the domain wall conserving CDW low-value entanglement

Middle is the smooth shift from (...1010...) to (...0101...) , the system don't know which to choose. Therefore is the superposition of these two configurations.

Kitaev-Hubbard 模型有很多研究的文章，有去找一个严格可解的模型的[2]，有去做RG的[3]，也有如本文做DMRG的[4]，得到的相图都与本文类似，都聚焦于描述基态能量，纠缠熵和占据数，对我的DMRG计算很有参考价值。更复杂和普适的文章用对称性分类[5][6]，近期我难以关注这些内容。



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- [5] Ari M, Turner Frank, Pollman Erez Ber, Topological phases of one-dimensional fermions: An entanglement point of view, Phys. Rev. B 83, 075102(2011)
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